$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \frac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$
$$= 10\,\frac{x^5}{5} - 2\tan x + C$$
$$= 2x^5 - 2\tan x + C \quad ✓$$

You should check this answer by differentiating it.

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

$\frac{\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta\,\sin\theta} = \cot\theta \cdot \frac{1}{\sin\theta}$

SOLUTION This indefinite integral isn't immediately apparent in Table 1, so we u trigonometric identities to rewrite the function before integrating:

$\frac{1}{\sin\theta} = \text{cosec}\,\theta$ $\qquad$ $= \cot\theta \cdot \text{cosec}\,\theta$

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$
$$= \int \csc\theta\,\cot\theta\,d\theta = -\csc\theta + C$$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

EXAMPLE 3 Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

$$\frac{x^4}{4} - 6\cdot\frac{x^2}{2}\Bigg|_{x=0}^{x=3}$$

$$\frac{x^4}{4} - 3x^2\Bigg|_0^3 = \left(\frac{81}{4} - 27\right) - (0 - 0)$$

$$= -6.75$$

EXAMPLE 5 Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

$$\frac{2t^2}{t^2} + \frac{t^2\sqrt{t}}{t^2} - \frac{1}{t^2}$$

$$= 2 + \sqrt{t} - \frac{1}{t^2}$$

$$= 2 + t^{1/2} - t^{-2}$$

$$\int (2 + t^{1/2} - t^{-2})\, dt = 2t + \frac{t^{3/2}}{3/2} - \frac{t^{-1}}{-1} + C$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t} + C$$

$$\int_0^1 x\, dx \qquad \int_0^1 t\, dt \qquad \int_0^1 y\, dy$$

EXAMPLE 5 Evaluate the integral $\int_1^3 e^x\, dx$.

$$= e^x \Big|_1^3 = e^3 - e^1.$$

EXAMPLE 7 Evaluate $\int_3^6 \frac{dx}{x}$.

$$= \ln|x| \Big|_3^6$$

$\ln a - \ln b = \ln \frac{a}{b}$

$$= \ln 6 - \ln 3 = \ln \frac{6}{3}$$

$$= \ln 2.$$

21–46 Evaluate the integral.

21. $\int_{-2}^{3} (x^2 - 3)\, dx$

22. $\int_1^2 (4x^3 - 3x^2 + 2x)\, dx$

23. $\int_{-2}^{0} \left(\frac{1}{2}t^4 + \frac{1}{4}t^3 - t\right) dt$

24. $\int_0^3 (1 + 6w^2 - 10w^4)\, dw$

25. $\int_0^2 (2x - 3)(4x^2 + 1)\, dx$

26. $\int_{-1}^{1} t(1 - t)^2\, dt$

27. $\int_0^{\pi} (5e^x + 3\sin x)\, dx$

28. $\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx$

29. $\int_1^4 \left(\frac{4 + 6u}{\sqrt{u}}\right) du$

30. $\int_0^1 \frac{4}{1 + p^2}\, dp$

31. $\int_0^1 x(\sqrt[3]{x} + \sqrt[4]{x})\, dx$

32. $\int_1^4 \frac{\sqrt{y} - y}{y^2}\, dy$

33. $\int_1^2 \left(\frac{x}{2} - \frac{2}{x}\right) dx$

34. $\int_0^1 (5x - 5^x)\, dx$

35. $\int_0^1 (x^{10} + 10^x)\, dx$

36. $\int_0^{\pi/4} \sec\theta \tan\theta\, d\theta$

37. $\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta$

ex. $\int_0^{\pi} (5e^x + 3\sin x)\, dx$

$= 5e^x - 3\cos x \Big|_0^{\pi}$

$= (5e^{\pi} - 3 \cdot \underbrace{\cos\pi}_{-1}) - (5e^0 - 3\underbrace{\cos 0}_{1})$

$= 5e^{\pi} + 3 - 5 + 3$

$= 5e^{\pi} + 1$

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx.$$

$$= \int \frac{-du/8}{\sqrt{u}} = -\frac{1}{8}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C$$

$$= -\frac{1}{8}\cdot 2\, u^{1/2} + C = -\frac{1}{4}\sqrt{(1-4x^2)} + FB \checkmark$$

$u = 1-4x^2$

$du = -8x\,dx$

$\frac{du}{-8} = x\,dx$

Fener Bahçe

FB = constant of integration

$$\int 1\cdot\sqrt{2x+1}\,dx$$

$$\frac{1}{2}\int \sqrt{u}\cdot du = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + FB$$

$$= \frac{1}{2}\cdot\frac{2}{3}(2x+1)^{3/2} + FB$$

$$= \frac{1}{3}(2x+1)^{3/2} + FB$$

$u = 2x+1$

$du = 2\,dx$

$\frac{du}{2} = dx$

$\int \sqrt{x}\,dx$

$\int \sqrt{u}\cdot du$

$\int \sqrt{w}\,dw$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= \int -\frac{du}{u} = -\int \frac{du}{u} = -\ln|u| + FB$$

$$= -\ln|\cos x| + FB \checkmark$$

$$= \ln|\cos x|^{-1} + FB$$

$$= \ln\frac{1}{|\cos x|} + FB$$

$$= \ln|\sec x| + FB$$

$\tan x = \frac{\sin x}{\cos x}$

~~$u = \sin x$~~

~~$du = \cos x\,dx$~~

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$\int \frac{dx}{x} = \ln|x|$

$\int \frac{du}{u} = \ln|u|$

$$\int e^{5x}\,dx$$

$$\frac{1}{5}\int e^{u}\cdot du$$

$$= \frac{1}{5}e^{u} + FB$$

$$= \frac{1}{5}e^{5x} + FB$$

$u = 5x$

$du = 5\,dx$

$\frac{du}{5} = dx$

$\int e^{x}\,dx = e^{x} + C$

$\int e^{u}\,du = e^{u} + C$

$\int e^{w}\,dw = e^{w} + C$

$\int \sin(3x)\,dx$ $\qquad u = 3x$ $\qquad \int \sin x\,dx = -\cos x + C$

$\frac{1}{3}\int \sin u \cdot du$ $\qquad du = 3\,dx$ $\qquad \int \sin u\,du = -\cos u + C$

$\frac{du}{3} = dx$

$= \frac{1}{3} - \cos u + FB = -\frac{1}{3}\cos(3x) + FB$

---

find the indefinite integral.

$\int x\sqrt{x-1}\,dx = \int (u+1)\cdot \underbrace{\sqrt{u}}_{u^{1/2}}\,du$

$u = x - 1$

$du = 1 \cdot dx$

$x = u + 1$

$= \int (u^{3/2} + u^{1/2})\,du$

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} +$ ALNasR Ronaldo

$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} +$ Al NASR

EXAMPLE 8 Evaluate $\int_{1}^{2} \frac{dx}{(3-5x)^2}$.

$u = 3 - 5x$

$du = -5\,dx$

$\frac{du}{-5} = dx$

$u = 3 - 5x$

$x = 1 \Rightarrow u = -2$

$x = 2 \Rightarrow u = -7$

$-\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du$

$= -\frac{1}{5}\,\frac{u^{-1}}{-1}\Big|_{-2}^{-7} = \frac{1}{5}\,\frac{1}{u}\Big|_{-2}^{-7}$

$= \frac{1}{5}\left[\frac{1}{-7} - \frac{1}{-2}\right] = \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right)$

$= \frac{1}{14}$

$\int_{-1}^{2} |x|\,dx =$ Let me kill MATH.

$= \int_{-1}^{2} |x|\,dx = \frac{x^2}{2}\Big|_{-1}^{2} = \ldots\ldots$

$x \le 0 \quad x \ge 0$

$-1 \quad 0 \quad 2$

$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x \le 0 \end{cases}$

$\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} \underbrace{|x|}_{-x}\,dx + \int_{0}^{2} \underbrace{|x|}_{x}\,dx$

$= -\int_{-1}^{0} x\,dx + \int_{0}^{2} x\,dx = \ldots\ldots$

$|2| = 2$

$|-3| = -(-3) = 3$

$$\int_{-1}^{2} |2x-1|\,dx =$$

$-(2x-1) = -2x+1 = 1-2x$

$2x-1 \geq 0$

$2x \geq 1$

$x \geq \frac{1}{2}$

$2x-1<0$ | $2x-1 \geq 0$

-1 $\quad 1/2 \quad$ 2

$$= \int_{-1}^{1/2} |2x-1|\,dx + \int_{1/2}^{2} |2x-1|\,dx$$

($|2x-1| = 1-2x$ on the first interval, $|2x-1| = 2x-1$ on the second)

$$= \int_{-1}^{1/2} (1-2x)\,dx + \int_{1/2}^{2} (2x-1)\,dx = \ldots\ldots$$

EXAMPLE 9 Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x=1 \Rightarrow u = \ln 1 = 0$

$x=e \Rightarrow u = \ln e = 1$

$$= \int_0^1 u \cdot du$$

$$= \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

---

87. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

$$\int_0^2 f(2x)\,dx$$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

$x=0 \Rightarrow u=0$

$x=2 \Rightarrow u=4$

$$= \frac{1}{2}\int_0^4 f(u)\,du$$

$$= \frac{1}{2} \cdot 10 = 5$$

$\int_0^4 f(x)\,dx = 10$

$\int_0^4 f(u)\,du = 10$

$\int_0^4 f(w)\,dw = 10$

---

88. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$. (exercise)

We abbreviate the name of this theorem as FTC1. In words, it says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

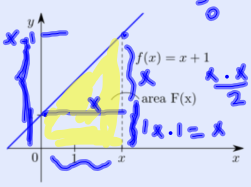
is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Let $f(x) = x + 1$. For any x, let $F(x)$ denote the area of the region under the graph of f from 0 to x:

$F(x) = \int_0^x (x+1)\,dx \quad F'(x) = x+1 = f$

$x + \frac{x^2}{2}$ = Area Function

Derivative of area Function $= 1 + x$



This region is composed of a triangle atop a rectangle, so we can use the familiar area formulas to find that $F(1) = 3/2$, $F(2) = 4$, and in general

$$\begin{aligned} F(x) &= \text{area of triangle} + \text{area of rectangle} \\ &= \tfrac{1}{2}x^2 + x. \end{aligned}$$

Of interest to us here is the observation that the derivative of this area function is equal to the original function:

$$F'(x) = x + 1 = f(x).$$

Example: Find $F'(x)$ if

$$F(x) = \int_3^x e^{t^2}\,dt. \quad \Rightarrow F'(x) = e^{x^2}.$$

Solution: Since $f(t) = e^{t^2}$ is a continuous function, the Fundamental Theorem of Calculus 1 tells us we can replace t by x in $f(t)$ to get

$$F'(x) = e^{x^2}.$$

Example: Find $G'(x)$ if

$$G(x) = \int_3^{x^2} e^{t^2}\,dt.$$

$H(u) = \int_3^u e^{t^2}\,dt$

$H'(u) = e^{u^2}$

$G(x) = H(x^2)$

$G'(x) = H'(x^2) \cdot 2x$ (outer derivative, inner derivative)

$G(x) = \int_3^x e^{t^2}\,dt$

$G'(x) = e^{x^2}$.

$H'(u) = e^{u^2}$

$H'(x^2) = e^{(x^2)^2} = e^{x^4}$

$= e^{x^4} \cdot 2x$ ✓

Area Problem

Figure 12 shows this approximation for $n = 2, 4, 8$, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.

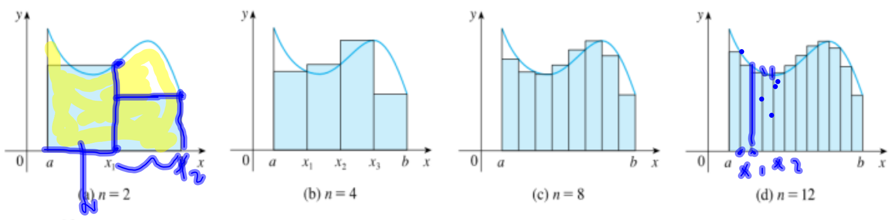


FIGURE 12

$f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots$

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

2 Definition of a Definite Integral If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Riemann Sum

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$

Riemann-Sum

integral

integration

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

$$\text{Area} = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

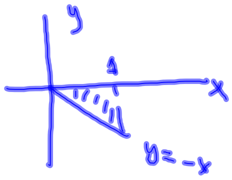
$0 < x < 5 \Rightarrow |x| = x$

$-7 < x < -3 \Rightarrow |x| = -x$

$-1 < x < 0 \Rightarrow |-x| = -x$

---

ex. Calculate the area under $y = -x$, $0 < x < 1$.



$$\int_0^1 -x\,dx = -\frac{x^2}{2}\Big|_0^1 = -\left(\frac{1}{2} - 0\right) = -\frac{1}{2}$$

$$\int_0^1 |-x|\,dx = \int_0^1 -(-x)\,dx$$

$|-5| = -(-5) = 5$

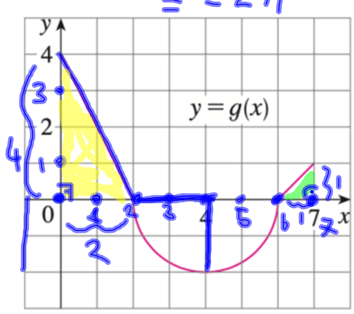
$$= \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} = \text{Area}$$

---

34. The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$ $= -2\pi$ (c) $\int_0^7 g(x)\,dx$

$\int_0^2 g(x)dx = 2 \cdot 4 \cdot \frac{1}{2} = 4$ ✓



b) $\pi r^2 = \pi \cdot 2^2 = 4\pi$

$4\pi / 2 = 2\pi$

c) $\int_0^7 g\,dx = \int_0^2 g\,dx + \int_2^6 g\,dx + \int_6^7 g\,dx$

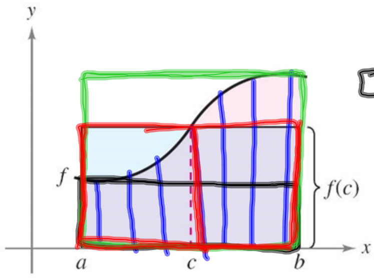
$4 + -2\pi + \frac{1}{2}$

$= \frac{9}{2} - 2\pi$ ✓

THEOREM 4.10 **Mean Value ~~Theorem~~ Rule for Integrals**

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$



$\square < \int f\,dx < \square$

Mean value rectangle:

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

EXAMPLE 2 Since $f(x) = 1 + x^2$ is continuous on the interval $[-1, 2]$, the Mean Value Theorem for Integrals says there is a number c in $[-1, 2]$ such that

$f(x) = 1 + x^2$

$f(c) = 1 + c^2$

$$\int_a^b f\,dx = f(c)(b-a)$$

$$\int_{-1}^{2} (1+x^2)\,dx = (1+c^2)\cdot 3$$

$$x + \frac{x^3}{3}\Big|_{-1}^{2} = 3(1+c^2)$$

$$\left(2+\frac{8}{3}\right) + \left(-1-\frac{1}{3}\right) = 3(1+c^2)$$

$$6 = 3(1+c^2) \Rightarrow 1 + c^2 = 2$$

$f(c) = 1 + c^2$

$c = -1, 1$

$f(1) = 2$

$f(-1) = 2$

$[-1, 2]$

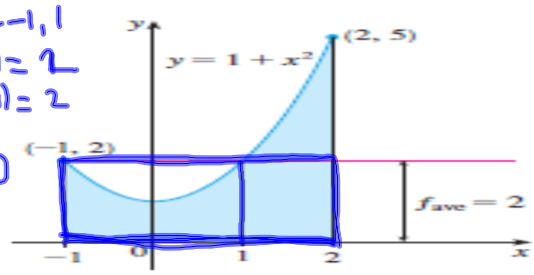


FIGURE 3

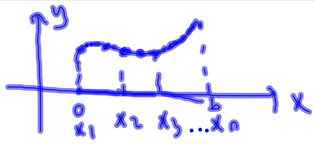
Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx. = \text{Average Value}$$

See Figure 4.32.

$$\frac{1}{b-a}\int_a^b f\,dx = f(c)\,\cancel{(b-a)}$$



$$\frac{f(x_1)+f(x_2)+\dots f(x_n)}{n}$$

$$\sum_{i=1}^{n} \frac{f(x_i)}{n}\cdot\frac{b-a}{b-a}$$

$$\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\frac{b-a}{n}\,\Delta x$$

$$\frac{1}{b-a}\int f\,dx$$

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

a b

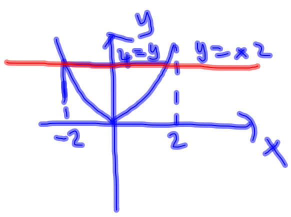
$$\frac{1}{b-a}\int_a^b f\,dx = \frac{1}{3}\int_1^4 (3x^2-2x)\,dx$$

$$= \frac{1}{3}\left(x^3 - x^2\right)\Big|_1^4$$

$$= \frac{1}{3}\left[(64-16) - (1-1)\right] \quad (48,\ 0)$$

$$= 16$$

Inverse trigonometric functions



You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test).

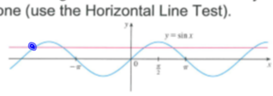


Figure 1

But the function $f(x) = \sin x$, $-\pi/2 \le x \le \pi/2$, *is* one-to-one (see Figure 2).

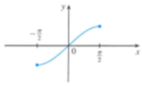


Figure 2 $y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

3

$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

$f: A \to B$

$f^{-1}: B \to A$

$\sin x: [-\pi/2, \pi/2] \to [-1, 1]$

$\arcsin x = \sin^{-1}(x) = [-1,1] \to [-\pi/2, \pi/2]$

Inverse Trigonometric Functions and Their Derivatives

The inverse function of this restricted sine function *f* exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

Since the definition of an inverse function says that

$$f^{-1}(x) = y \quad \Longleftrightarrow \quad f(y) = x$$

we have

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Thus, if $-1 \le x \le 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is *x*.

4

Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan(\arcsin \frac{1}{3})$.

a) $\sin^{-1}(\frac{1}{2}) = x$

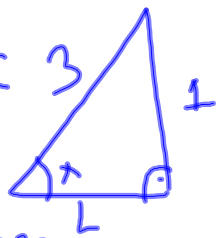
$\sin x = \frac{1}{2}$, $-\pi/2 < x < \pi/2$

$x = 30° = \frac{\pi}{6}$

b) $\tan(\arcsin \frac{1}{3})$ = $\tan x = \frac{1}{L} = \frac{1}{2\sqrt{2}}$

$\arcsin \frac{1}{3} = x$

$\sin x = \frac{1}{3}$



$L^2 + 1^2 = 9$

$L^2 = 8$ $L = 2\sqrt{2}$

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

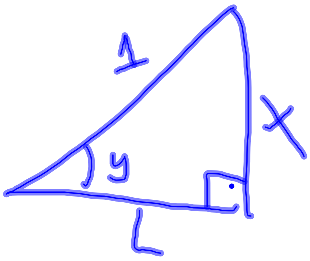
Let $y = \sin^{-1}x$. Then $\sin y = x$ and $-\pi/2 \le y \le \pi/2$.

$$y = \sin^{-1}(x) \Rightarrow \frac{dy}{dx} = ?$$

$$\sin y(x) = x$$

Differentiate

$$\cos y \cdot y' = 1 \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$



$$L^2 + x^2 = 1$$

$$L^2 = 1 - x^2$$

$$L = \sqrt{1-x^2}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$(\arcsin u(x))' = u' \cdot \frac{1}{\sqrt{1-u^2}}$$

4.5 Derivatives of Inverse Trigonometric Functions

(page261-267)

$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} u'$ inner derivative
$\frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\cos^{-1} u\right] = -\frac{1}{\sqrt{1-u^2}} u'$ inner
$\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$	$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\cdot u'$
$\frac{d}{dx}\left[\csc^{-1} x\right] = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\csc^{-1} u\right] = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\cdot u'$
$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\cdot u'$
$\frac{d}{dx}\left[\cot^{-1} x\right] = -\frac{1}{1+x^2}$	$\frac{d}{dx}\left[\cot^{-1} u\right] = -\frac{1}{1+u^2}\cdot u'$

$\frac{d}{dx}\left[\ln(\tan^{-1}(x))\right] = \frac{u'}{u}$ (with $u = \tan^{-1}(x)$) $\frac{d}{dx}\ln(u) = \frac{u'}{u}$

$= \frac{\frac{1}{1+x^2}}{\tan^{-1}(x)} = \frac{1}{\tan^{-1}(x)\cdot(1+x^2)}$

$\int \frac{1}{(1+x^2)\arctan x}dx = \int \frac{1}{u} du$ $u = \arctan x$, $du = \frac{1}{1+x^2}dx$

$= \ln|u| + C = \ln|\arctan x| + C$

---

7. $\frac{d}{dx}\left[\tan^{-1}(\pi x)\right]$ (with $u = \pi x$) $= \left[\tan^{-1}(u)\right]'$

$= \frac{u'}{1+u^2} = \frac{\pi}{1+\pi^2 x^2}$

$$\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$$

$u = \sin^{-1} x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$$\int \frac{1}{u}\cdot du = \ln|u| + C$$

$$= \ln\left|\sin^{-1}(x)\right| + FB_{istanbul}$$

<u>Next time</u>: 6.1

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution:

The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$.

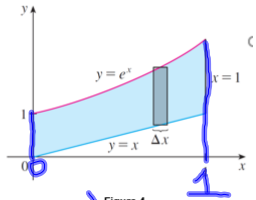


Figure 4

Set-up integral (calc2)

$$\int_0^1 (e^x - x)\,dx.$$

$$\text{Area} = e^x - \frac{x^2}{2}\Big|_0^1 = \left[e^1 - \frac{1}{2}\right] - \left[e^0 - 0\right] = e - \frac{1}{2} - 1 = e - \frac{3}{2} > 0$$

(1 − 0)

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Set-up the integral

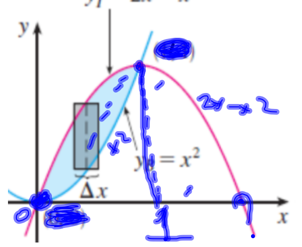
$$\int_0^1 \left[(2x - x^2) - (x^2)\right] dx$$

$$2x - x^2 - x^2$$

$$= \int_0^1 (2x - 2x^2)\,dx$$

$$2(x - x^2)$$

$$= 2\int_0^1 (x - x^2)\,dx$$



$2x - x^2 = x^2$, solve for x

$2x - 2x^2 = 0$

$2x(1 - x) = 0$

$x = 0 \quad x = 1$

$$\text{Area} = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)\right]$$

$$= 2 \cdot \frac{1}{6} = \frac{1}{3}$$

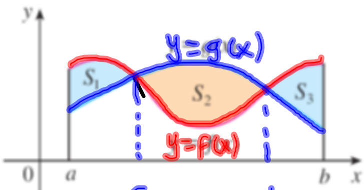
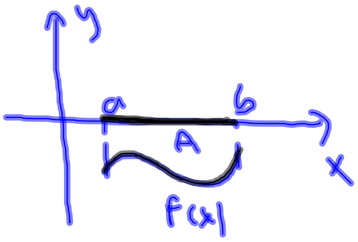


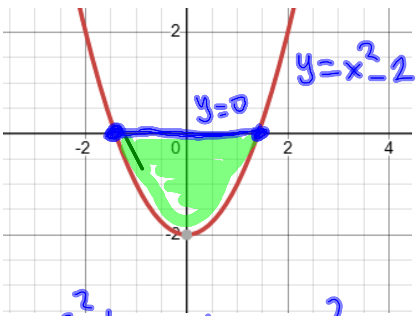
Figure 11

Area $= \int_a^c (f-g)\,dx + \int_c^d (g-f)\,dx + \int_d^b (f-g)\,dx = \int_a^b |f-g|\,dx$



$A = \int_a^b |0-f|\,dx = \int_a^b |f|\,dx$

$|-f| = |f|$



Set-up:

$\int_{-2}^{2} \left[0-(x^2-2)\right]dx$

$\int_{-2}^{2} (2-x^2)\,dx$

$\int_{-2}^{2} |\underbrace{x^2-2}_{\leq 0}|\,dx = \int_{-2}^{2} (2-x^2)\,dx$

Area $= 2x - \frac{x^3}{3}\Big|_{-2}^{2}$

$= \left(4-\frac{8}{3}\right) - \left(-4+\frac{8}{3}\right)$

$= 4-\frac{8}{3}+4-\frac{8}{3}$

$= 8-\frac{16}{3} = \frac{8}{3}.$

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution:

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \le x \le \pi/2$). The region is sketched in Figure 12. Observe that $\cos x \ge \sin x$ when $0 \le x \le \pi/4$ but $\sin x \ge \cos x$ when $\pi/4 \le x \le \pi/2$.

$\sin x = \cos x$

$x = \pi/4 = 45^\circ$

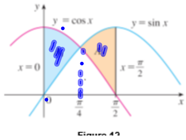


Figure 12

Set-up:

$$\int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$\text{Area} = \sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$$= \left[\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - (\sin 0 + \cos 0)\right] + \left[\left(-\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right) - \left(-\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\right)\right]$$

($\sin\frac{\pi}{4} = \sqrt{2}/2$, $\cos\frac{\pi}{4} = \sqrt{2}/2$, $\sin 0 = 0$, $\cos 0 = 1$, $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$)

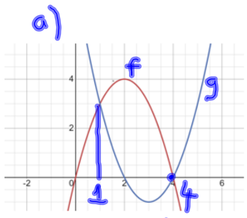
$$= 2\sqrt{2}$$

Find the area between the two curves

a. $f(x) = 4x - x^2$ and $g(x) = x^2 - 6x + 8$

b. $f(x) = 2x + x^2 - x^3$ and $g(x) = 0$

a)



$x^2 - 6x + 8 = 4x - x^2$

$x^2 - 6x + 8 - 4x + x^2 = 0$

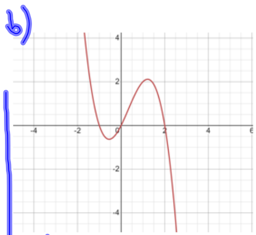
$2x^2 - 10x + 8 = 0$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4 \quad x = 1$

b)

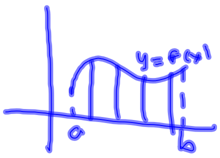


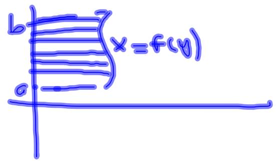
(exercise)

$$\text{Area} = \int_1^4 \left[(4x - x^2) - (x^2 - 6x + 8)\right] dx = \int_1^4 \left[-2x^2 + 10x - 8\right] dx$$

$4x - x^2 - x^2 + 6x - 8$

= (exercise) – Calculate



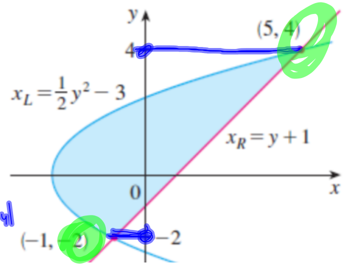


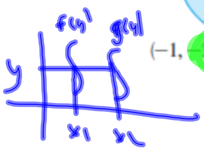
$x = y+1$

EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

$y^2 - 6 = 2x$

$\frac{y^2}{2} - 3 = x$





Set-up:

$$\int_{-2}^{4}\left[(y+1)-\left(\frac{y^2}{2}-3\right)\right]dy$$

$y+1-\frac{y^2}{2}+3$

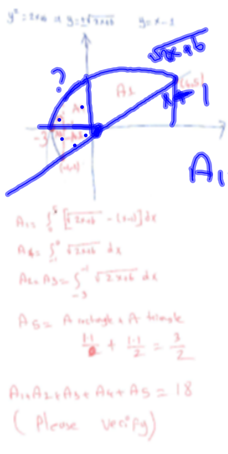
$$\int_{-2}^{4}\left(y-\frac{y^2}{2}+4\right)dy$$

$=$ (exercise)

$y^2 = 2x+6$

$y = \sqrt{2x+6}$

If you want to integrate for x



$$A_1 = \int_0^4 \left[\sqrt{2x+6} - (x-1)\right]dx$$

Next time: 6.2

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

The Pyramid Arena in Memphis

EXAMPLE 2.1 Computing Volume from Cross-Sectional Areas

The Pyramid Arena in Memphis (pictured in the margin) has a square base of side approximately 600 feet and a height of approximately 320 feet. Find the volume of the pyramid with these measurements.

Solution To use (2.1), we need to have a formula for the cross-sectional area. The square horizontal cross sections of the pyramid make this easy, but we need a formula for the size of the square at each height. Orient the x-axis upward through the point at the top of the pyramid. At $x = 0$, the cross section is a square of side 600 feet. At $x = 320$, the cross section can be thought of as a square of side 0 feet. If $f(x)$ represents the side length of the square cross section at height x, we know that $f(0) = 600$, $f(320) = 0$ and $f(x)$ is a linear function. (Think about this; the sides of the pyramid do not curve.) The slope of the line is $m = \frac{600 - 0}{0 - 320} = -\frac{15}{8}$ and we use the y-intercept of 600 to get

$$f(x) = -\frac{15}{8}x + 600.$$

Since this is the length of a side of a square, the cross-sectional area is simply the square of this quantity. Then from (2.1), we have

$$V = \int_0^{320} A(x)\,dx = \int_0^{320} \left(-\frac{15}{8}x + 600\right)^2 dx.$$

Observe that we can evaluate this integral by substitution, by taking $u = -\frac{15}{8}x + 600$, so that $du = -\frac{15}{8}\,dx$. This gives us

$$V = \int_0^{320} \left(-\frac{15}{8}x + 600\right)^2 dx = -\frac{8}{15}\int_{600}^{0} u^2\,du$$
$$= \frac{8}{15}\int_0^{600} u^2\,du = \frac{8}{15}\frac{u^3}{3}\Big|_0^{600} = 38{,}400{,}000\ \text{ft}^2.$$

$\int A(x)dx$

$f(x) \to$ side length

height

$f(0) = 600$

$f(320) = 0$ (exercise) slope

$(0, 600), (320, 0)$ f is a linear function

$(\text{side length})^2$ (exercise)

$V = \int_0^{320} \left(-\frac{15}{8}x + 600\right)^2 dx$

$u = -\frac{15}{8}x + 600$

$du = -\frac{15}{8} \cdot dx$ — (exercise)

$V = \frac{1}{3} \times B \times h$

$V = \frac{1}{3} \times lw \times h$

$V = \frac{1}{3}\,lwh$

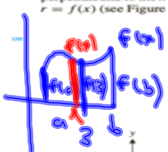
$V = \int A(x)dx$

We can calculate the volume by this formula. Give it a shot !!!

$V = \frac{1}{3} \cdot 600 \cdot 600 \cdot 320 = 38{,}400{,}000$

The Method of Disks

Suppose that $f(x) \ge 0$ and f is continuous on the interval $[a, b]$. Take the region bounded by the curve $y = f(x)$ and the x-axis, for $a \le x \le b$, and revolve it about the x-axis, generating a solid (see Figures 5.16a and 5.16b). We can find the volume of this solid by slicing it perpendicular to the x-axis and recognizing that each cross section is a circular disk of radius $r = f(x)$ (see Figure 5.16b). From (2.1), we then have that the volume of the solid is



$$V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{cross-sectional area} = \pi r^2} dx. \qquad (2.2)$$

$V = \int A(x)dx$

$= \int \pi r^2 dx$

$= \pi \int f^2(x)dx$

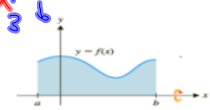


FIGURE 5.16a
$y = f(x) \ge 0$

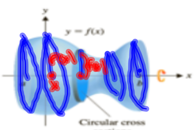
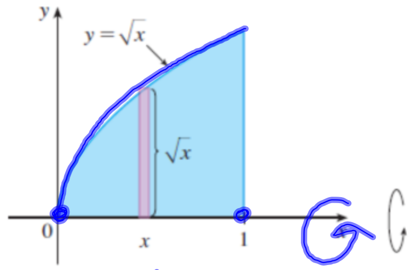
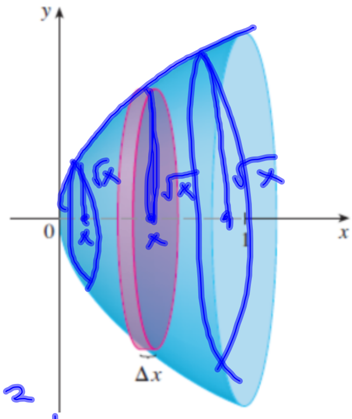


FIGURE 5.16b
Solid of revolution

Since the cross sections of such a solid of revolution are all disks, we refer to this method of finding volume as the **method of disks.**

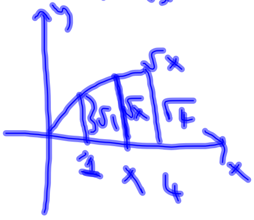
EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.





$V = \pi \int_0^1 r^2 \, dx$

$r = \sqrt{x}$



$V = \pi \int_0^1 (\sqrt{x})^2 \, dx$

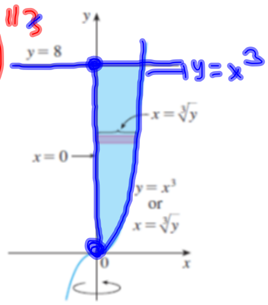
$= \pi \int_0^1 x \, dx = \pi \frac{x^2}{2} = \pi\left(\frac{1}{2} - 0\right)$

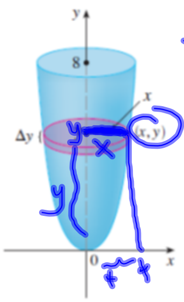
$= \pi/2$.

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

$y^{1/3} = (x^3)^{1/3}$

$y^{1/3} = x$





$V = \int \pi r^2 \, dy$

$= \pi \int_0^8 x^2 \, dy$

$= \pi \int_0^8 (y^{1/3})^2 \, dy$

$(y^{1/3})^2 = y^{2/3}$ $\quad V = \pi \int_0^8 y^{2/3} \, dy$.

$8^{5/3} = (2^3)^{5/3} = 2^5 = 32$

$= \pi \frac{3}{5} y^{5/3} \Big|_0^8 = \frac{3\pi}{5}\left(8^{5/3} - 0\right)$, with $8^{5/3} = 32$

$= \frac{96}{5}\pi$.

EXAMPLE 2.6 Computing Volumes of Solids with and without Cavities

Let R be the region bounded by the graphs of $y = \frac{1}{4}x^2$, $y = 0$ and $y = 1$. Compute the volume of the solid formed by revolving R about (a) the y-axis, (b) the x-axis and (c) the line $y = 2$. d) around $x = 2$.

$4y = x^2$

$\sqrt{4y} = x$

Solution (a) The region R is shown in Figure 5.20a and the solid formed by revolving it about the y-axis is shown in Figure 5.20b. Notice that this part of the problem is similar to example 2.5.

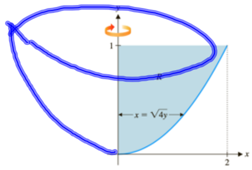


FIGURE 5.20a
$x = \sqrt{4y}$

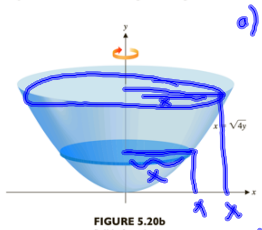


FIGURE 5.20b
Solid of revolution

From (2.3), the volume is given by

$$V = \int_0^1 \underbrace{\pi\left(\sqrt{4y}\right)^2}_{\pi r^2} dy = \pi \frac{4}{2} y^2 \Big|_0^1 = 2\pi.$$

a) $\pi \int_0^1 r^2 \, dy$

$\pi \int_0^1 x^2 \, dy$

$= \pi \int_0^1 4y \, dy$

$= 4\pi \frac{y^2}{2} \Big|_0^1$

$= \frac{4\pi}{2}(1 - 0)$

$= 2\pi$

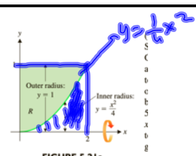


FIGURE 5.21a
$y = \frac{1}{4}x^2$

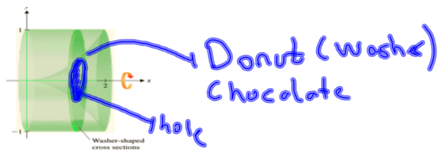
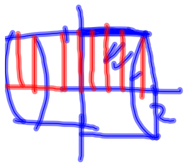


FIGURE 5.21b
Solid with cavity

$$V = \pi \int_0^2 1^2 \, dx - \pi \int_0^2 \left(\frac{1}{4}x^2\right)^2 dx$$



$$V = \pi \int_0^2 1 \, dx - \frac{\pi}{16} \int_0^2 x^4 \, dx$$

$\frac{x^5}{5} \Big|_0^2 = \frac{32}{5}$

$= 2\pi - \frac{\pi}{16} \cdot \frac{32}{5}$

$= 2\pi - \frac{32}{80}\pi = \frac{128}{80}\pi = \frac{8}{5}\pi$

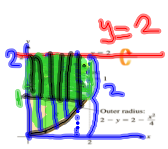


FIGURE 5.22a
Revolve about $y = 2$

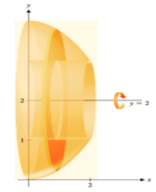
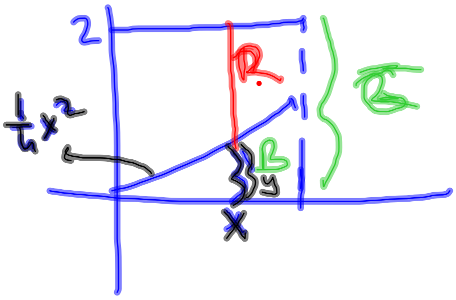


FIGURE 5.22b
Solid of revolution

G	Integral
x (y=0)	x
y (x=0)	y
y = 2	x
x = 2	y

$$V = \pi \int (2 - \tfrac{1}{4}x^2)^2 dx - \pi \int 1^2 \, dx$$



$R = G - B$

$= 2 - \frac{1}{4}x^2$

$$V = \pi \int_0^2 \left(2 - \frac{1}{4}x^2\right)^2 dx - \pi \int_0^2 1^2 dx$$

$x\Big|_0^2 = 2$

$(a-b)^2 = a^2 - 2ab + b^2$, with $a = 2$, $b = \frac{1}{4}x^2$

$$\left(2 - \frac{1}{4}x^2\right)^2 = 4 - 2 \cdot 2 \cdot \frac{1}{4}x^2 + \frac{1}{16}x^4$$

$$= 4 - x^2 + \frac{1}{16}x^4$$

$$\int_0^2 \left(4 - x^2 + \frac{1}{16}x^4\right) dx = 4x - \frac{x^3}{3} + \frac{1}{80}x^5 \Big|_0^2$$

$$= 8 - \frac{8}{3} + \frac{32}{80}$$

$$= \frac{86}{15}$$

Final: $\frac{86}{15}\pi - 2\pi = \frac{56}{15}\pi$

---

d) $x = 2$ (rotate) : exercise.

Questions

a) Set-up the integral

b) Calculate

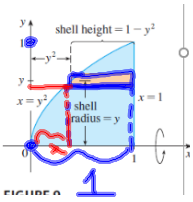
- Example 6

Use cylindrical shells to find the volume of the solid obtained by rotating about the *x*-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

$y = \sqrt{x}$

$y^2 = x$

Shell: $\int_0^1 2\pi y \text{ height } dy$

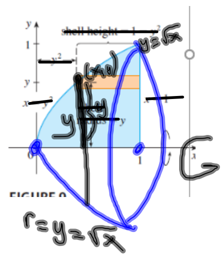


$\int_0^1 2\pi y (1 - y^2)\, dy$

$2\pi \int_0^1 (y - y^3)\, dy = 2\pi \left. \frac{y^2}{2} - \frac{y^4}{4} \right|_0^1$

$= 2\pi \left(\frac{1}{2} - \frac{1}{4}\right) = 2\pi \frac{1}{4} = \pi/2$

Disk:



$r = y = \sqrt{x}$

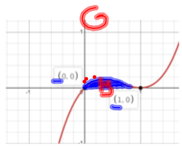
$V = \pi \int_0^1 r^2\, dx$

$= \pi \int_0^1 (\sqrt{x})^2\, dx = \pi \left. \frac{x^2}{2} \right|_0^1$

$= \frac{\pi}{2}$

Find the volume of the solid obtained by rotating about the *y*-axis the region bounded by $y = x(x-1)^2$ and ~~$y = 0$.~~ x-axis

Shell: $\int_0^1 2\pi x \text{ height } dx$



$V = 2\pi \int_0^1 x \; x(x-1)^2\, dx$

$(x-1)^2 = x^2 - 2x + 1$

$(a-b)^2 = a^2 - 2ab + b^2$

$V = 2\pi \int_0^1 x^2 (x-1)^2\, dx$

$V = 2\pi \int_0^1 x^2 (x^2 - 2x + 1)\, dx = 2\pi \int_0^1 (x^4 - 2x^3 + x^2)\, dx$

$= 2\pi \left. \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} \right|_0^1$

$= 2\pi \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = 2\pi \cdot \frac{1}{30} = \frac{1}{15}\pi.$

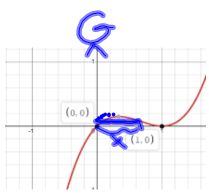
$\frac{1}{30}$

$y = x(x-1)^2 = x^3 - 2x^2 + 1$

$x^2 - 2x + 1$

$y = x^3 - 2x^2 + x$

$x = ?$

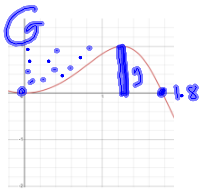


Disk:

$V = \int (x)^2\, dy - \int (\quad)^2\, dy$

Find the volume of the solid obtained by rotating y=sin(x²) around the *y*-axis the region bounded by *y* = 0.

Shell: $\int_0^{1.8} 2\pi x \text{ height } dx$



$V = 2\pi \int_0^{1.8} x \sin(x^2)\, dx$

$u = x^2$

$du = 2x\, dx$

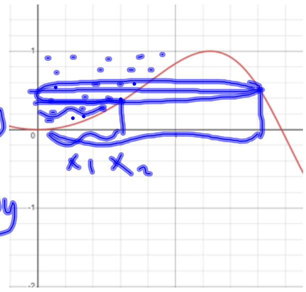
$V = \pi \int_0^{(1.8)^2} \sin u \cdot du$

$= \pi \left\{ -\cos u \Big|_0^{(1.8)^2} \right\} = \text{(exercise)}$

$y = \sin(x^2)$

$\arcsin y = x^2$

Disk:



$V = \pi \int_0^1 (x)^2 dy - \pi \int (\ )^2 dy$

$\pi \int_0^1 \arcsin y\, dy - \pi \int (\ )^2 dy$

?

STOP

Example 7

Find the volume of the solid $y = x^{3/2}$ and *y* = 8 and x=0 obtained by rotating around x-axis

$y^{2/3} = (x^{3/2})^{2/3}$

$y^{2/3} = x$

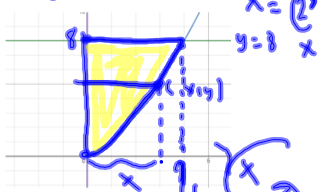
$(x^{3/2})^{2/3} = (8)^{2/3}$

$x = (2^3)^{2/3}$

$y = 8 \quad x = 4$

Shell: $\int_0^8 2\pi y \text{ height } dy$

$x = y^{2/3}$



$2\pi \int_0^8 y^{5/3} dy$

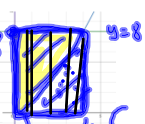
$2\pi \frac{3}{8} y^{8/3} \Big|_0^8$

$2\pi \frac{3}{8}(8^{8/3}) = 2\pi \frac{3}{8} \cdot 256$

$= 192\pi$

$8^{8/3} = (2^3)^{8/3} = 256$

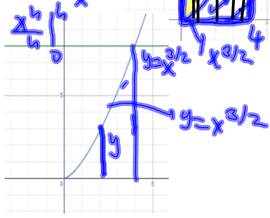
Disk: $V = \pi \int_0^4 (8)^2 dx - \pi \int_0^4 (x^{3/2})^2 dx$



$V = 64\pi \cdot 4 - 64\pi \qquad x \Big|_0^4 \qquad \frac{x^4}{4}\Big|_0^4$

$= 256\pi - 64\pi$

$= 192\pi$



Find the volume of the solid $y=x-x^2$ and $y = 0$ obtained by rotating around x=2 (Shell method)

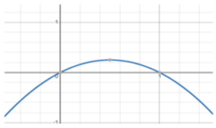
Shell:

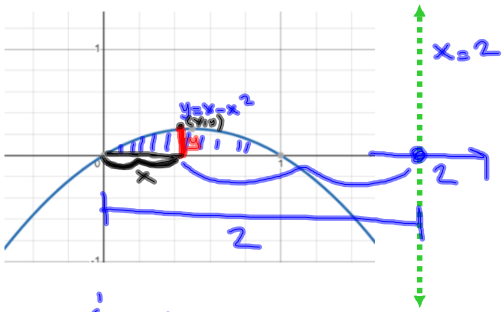
Rotate x: (y=0) $\int 2\pi y \text{ height } dy$

Rotate y: (x=0) $\int 2\pi x \text{ height } dx$

Rotate y=a $\int 2\pi$ distance between axis of rotation and sample rectangle height dy

Rotate x=b $\int 2\pi$ distance between axis of rotation and sample rectangle height dx





$V=\int_0^1 2\pi$ distance between axis of rotation and sample rectangle height dx

$V=\int_0^1 2\pi(2-x)(x-x^2)dx$. Set-up.

Quiz 1 : 10+1=11 mcqs. 11x1=11 pts.

5.5
6.1
6.2
6.3

1. Set-up integral to calculate area .....
   Calculate area .... (6.1)
2. Set-up integral to calculate volume by disk (or shell) (6.2, 6.3)
   calculate volume ... by disk (or shell)
3. 5.5: Substitution integrals: $\int$..... calculate
   Average value
   Fundamental theorem of Calc.
   $g(x)=\int_0^x f(t)dt$
   $g'(x)=f(x)$
   inverse trig. functions.

SLIDES
CLASS NOTES
Problems in the syllabus

Shell:

Rotate x ($y=0$) : $\int 2\pi\, y$ height dy

Rotate y ($x=0$) : $\int 2\pi\, x$ height dx

Rotate $y=b$: $\int 2\pi$ (Distance between axis of rotation and rectangle) height dy

Rotate $x=a$: $\int 2\pi$ (Distance between axis of rotation and rectangle) height dx

---

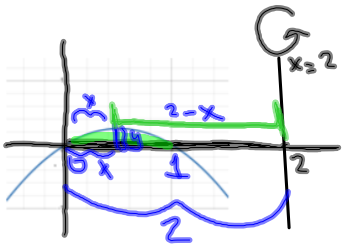
Find the volume of the solid $y=x-x^2$ and $y = 0$ obtained by rotating around $x=2$ Set-up by Shell.

See the class notes for the solution

Set-up integra

$\int_0^1 2\pi \, (2-x) \,$ height dx, height $= (x-x^2)$

$$2\pi \int_0^1 (2-x)(x-x^2)\,dx$$



5.5, 6.1, 6.2, 6.3 ⟹ Quiz Lessons.

7.1: $\int xe^x\,dx$, 7.2 $\int \sin^7 x \cos^3 x\,dx$

7.3 $\int \sqrt{1-x^2}\,dx$ 7.4 $\int \frac{x^7}{x^4+1}\,dx$.

7.1 : Integration by parts.

$\int (fg)' dx = \int f' \cdot g\,dx \; \int f \cdot g'\,dx$

$$\int f g'\,dx = fg - \int f' g\,dx$$

$f \xrightarrow{\text{dif.}} f'$

$g' \xrightarrow{\text{int.}} g$

$\int f g'\,dx = fg - \int g f'\,dx$

I. Guidelines for Selecting u and dv: (f and g')

(There are always exceptions, but these are generally helpful.)

"L-I-A-T-E" Choose 'u' (f) to be the function that comes first in this list:

L: Logrithmic Function

I: Inverse Trig Function

A: Algebraic Function

T: Trig Function

E: Exponential Function

$$\int \underbrace{x}_{f} \underbrace{\sin x}_{g'} \, dx =$$

$f = x \xrightarrow{\text{dif.}} f' = 1$

$g' = \sin x \xrightarrow{\text{int.}} g = -\cos x$

$$= fg - \int g f' \, dx$$

$$= -x\cos x - \int -\cos x \cdot 1 \, dx$$

$$= -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + \text{ALNASR}$$

$$\int \underbrace{x}_{f} \underbrace{e^x}_{g'} \, dx = fg - \int g f' \, dx$$

$f = x \longrightarrow f' = 1$

$g' = e^x \longrightarrow g = e^x$

$$= xe^x - \int e^x \cdot 1 \, dx$$

$$= xe^x - e^x + \text{TALISCA}$$

$$= e^x(x-1) + \text{TALISCA}$$

$$\int \ln x \, dx = \frac{1}{x} + c \quad \text{(Common mistake)}$$

$$\int \ln x \, dx = \int \underbrace{\ln x}_{f} \cdot \underbrace{1}_{g'} \, dx$$

$f = \ln x \longrightarrow f' = \frac{1}{x}$

$g' = 1 \longrightarrow g = x$

$$= fg - \int g f' \, dx$$

$$= x\ln x - \int x \cdot \frac{1}{x} \, dx = x\ln x - x + \text{FenerBahçe}$$

$$I = \int \underbrace{x^2}_{f} \underbrace{\sin x}_{g'} \, dx =$$

$f = x^2 \longrightarrow f' = 2x$

$g' = \sin x \longrightarrow g = -\cos x$

$$= fg - \int g f' \, dx$$

$$= -x^2\cos x - 2\int -\cos x \cdot x \, dx$$

$$= -x^2\cos x + 2\int \underbrace{x}_{f} \underbrace{\cos x}_{g'} \, dx \longrightarrow$$

$= fg - \int g f' \, dx$

$f = x \longrightarrow f' = 1$

$g' = \cos x \longrightarrow g = \sin x$

$$= -x^2\cos x + 2\left[x\sin x - \int \underbrace{\sin x}_{g} \cdot \underbrace{1}_{f'} \, dx\right]$$

$$= -x^2\cos x + 2\left[x\sin x + \cos x\right] + \text{TALISCA}$$

ex. $\int x^{7} \cos x \, dx$. Integration by parts is applied ...... times

a) 7 (circled) b) $\frac{1}{7}$ c) 77 d) $\frac{1}{77}$ e) 1907

$$\int \arctan x\,dx = \int \underbrace{\arctan x}_{f} \cdot \underbrace{1}_{g'}\, dx =$$

$f = \arctan x$

$f' = \frac{1}{1+x^2}$.

$g' = 1$

$g = x$

$$= fg - \int g f' dx$$

$$= x\arctan x - \int x \cdot \frac{1}{1+x^2} dx$$

$u = 1+x^2$ $\quad du = 2x\,dx$ $\quad \frac{du}{2} = x\,dx$

$$\int \frac{du/2}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + \text{Ronaldo}.$$

Final answer.

$[\ln(u)]' = u' \cdot \frac{1}{u} = \frac{u'}{u}$

Example B: $\int \underbrace{\sin x}_{g'} \underbrace{\ln(\cos x)}_{f}\, dx$

$f = \ln(\cos x)$ ($u = \cos x$)

$f' = -\frac{\sin x}{\cos x}$

$g' = \sin x \Rightarrow g = -\cos x$

$$= fg - \int g f' dx$$

$$= -\cos x \cdot \ln(\cos x) - \int -\cos x \cdot -\frac{\sin x}{\cos x}\, dx$$

$$= -\cos x \ln(\cos x) - \int \sin x\, dx$$

$$= -\cos x \ln(\cos x) + \cos x + \text{TALISCA}.$$

$$\int x e^x dx = x e^x - e^x + C$$ → We solved it.

$$\int \underbrace{x^2}_{f} \underbrace{e^x}_{g'} = fg - \int g f' dx$$

$f = x^2 \to f' = 2x$

$g' = e^x \to g = e^x$

$$= x^2 e^x - 2\int e^x \cdot x\, dx$$ (int. by parts)

$$= x^2 e^x - 2[x e^x - e^x] + \text{Talisca}$$

$$= x^2 e^x - 2x e^x + 2e^x + \text{Talisca}$$

$$= e^x(x^2 - 2x + 2) + \text{Talisca}$$

example. How many times integration by parts is applied to solve $\int x^{1907} e^x dx$.

a) 1 (b) 1907 c) 7 d) 10 e) 2024

Evaluate $\int e^x \sin x \, dx$. $= I$

(e^x labelled g', $\sin x$ labelled f)

$f = \sin x \to f' = \cos x$

$g' = e^x \to g = e^x$

$I = fg - \int g f' \, dx$

$= e^x \sin x - \int e^x \cos x \, dx$ (e^x labelled g', $\cos x$ labelled f) $= fg - \int g f' \, dx$

$f = \cos x \to f' = -\sin x$

$g' = e^x \to g = e^x$

$= e^x \sin x - \left[e^x \cos x - \int e^x \cdot - \sin x \, dx \right]$

$I = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$ (the last integral labelled I)

$I = e^x(\sin x - \cos x) - I$

$2I = e^x(\sin x - \cos x)$

$I = \frac{1}{2} e^x (\sin x - \cos x) + \text{TAUSCA}$

Monday: 7.1 and 7.2 ✓

Wednesday: Quiz day ✓

9 - 10.15 S1.36

$$\int_a^b f g' dx = fg\Big|_a^b - \int_a^b g f' dx$$

$$\int_0^1 x e^x dx = xe^x\Big|_0^1 - \int_0^1 \underbrace{e^x \cdot 1 dx}_{e^x} \qquad f = x \to f' = 1, \quad g' = e^x \to g = e^x$$

$$= (1 \cdot e^1 - 0 e^0) - \left(e^x\Big|_0^1\right)$$

$$= e - [\underbrace{e^1}_{e} - \underbrace{e^0}_{1}]$$

$$= e - e + 1 = 1.$$

7.2: Trigonometric integral

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin^2 x \, dx = ? \quad \int \cos^2 x \, dx = ?$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\boxed{\sin^2 x = \frac{1 - \cos 2x}{2}} \checkmark$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

$$\cos 2x = \cos^2 x - 1 + \cos^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\boxed{\cos^2 x = \frac{1 + \cos 2x}{2}} \checkmark$$

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x) dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + \text{ALNASSR}$$

$$\int \cos 2x \, dx \qquad u = 2x, \quad du = 2dx, \quad \frac{du}{2} = dx \qquad \Rightarrow \frac{1}{2}\int \cos u \, du = \frac{1}{2}\sin u$$

$$\int \cos^2 x \, dx = \frac{1}{2}\int (1 + \cos 2x) dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + \text{TALISCA}$$

$$\int \sin x \cdot \cos x \, dx = \int u \, du \qquad u = \sin x, \quad du = \cos x \, dx$$

$$= \frac{u^2}{2} + \text{TALISCA}$$

$$= \frac{\sin^2 x}{2} + \text{TALISCA}$$

$$\int \sin^2 x \cdot \cos^2 x \, dx =$$

$$= \frac{1}{8}\int (1-\cos 4x)\,dx$$

$$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + \text{Talisca}$$

$a^2 - b^2 = (a-b)\cdot(a+b)$

$$\sin^2 x \cos^2 x = \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2}$$

$$= \frac{1}{4}\left(1 - \cos^2 2x\right)$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\cos^2 x \cdot \sin^2 x = \frac{1}{4}\left[1 - \left(\frac{1+\cos 4x}{2}\right)\right]$$

$$= \frac{1}{8}(1-\cos 4x)$$

$$-\frac{1}{2} + \frac{\cos 4x}{2}$$

$$\frac{1}{2} - \frac{1}{2}\cos 4x$$

$$\frac{1}{2}(1-\cos 4x)$$

$$\int \sin^3 x \, dx = \int \underbrace{\sin^2 x}_{(1-\cos^2 x)} \cdot \underbrace{\sin x \, dx}_{-du}$$

$$= -\int (1-u^2)\,du$$

$$= -\left(u - \frac{u^3}{3}\right) + \text{Talisca} =$$

$$= -\cos x + \frac{\cos^3 x}{3} + \text{Talisca}$$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dv$

$\cos^2 x + \sin^2 x = 1$

$$\int \cos^3 x \, dx = \int \underbrace{\cos^2 x}_{(1-\sin^2 x)} \underbrace{\cos x \, dx}_{du}$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int (1-u^2)\,du = u - \frac{u^3}{3} + \text{FenerBahçe}$$

$$= \sin x - \frac{\sin^3 x}{3} + \text{Fenerbahçe}$$

$$\int \sin^3 x \cdot \cos^3 x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \underbrace{\sin^3 x}_{u^3} \cdot \underbrace{\cos^2 x}_{(1-\sin^2 x) = (1-u^2)} \cdot \underbrace{\cos x \, dx}_{du} =$$

$$= \int (u^3 - u^5)\,du = \frac{u^4}{4} - \frac{u^6}{6} + \text{Ronaldo}$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + \text{Ronaldo}$$

Professor,

We love you so much, but we have a question?
Why you did not save a sine factor?
Please, answer!

$$\int \cos^3 x \cdot \sin^3 x \, dx = \int \underbrace{\cos^3 x}_{u^3} \cdot \underbrace{\sin^2 x}_{(1-\cos^2 x) = (1-u^2)} \cdot \underbrace{\sin x \, dx}_{-du}$$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x$

$$= \int (-u^3 + u^5)\,du = -\frac{u^4}{4} + \frac{u^6}{6} + \text{Ronaldo}$$

$$= -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + \text{Ronaldo}$$

Wednesday: 9 – 11

12 – 1 Quiz 1 Test

Next week: 7.2 and 7.3

— –

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

$\sin^6 x = (\sin^2 x)^3 = (1-\cos^2 x)^3$

$\sin^7 x = (\sin^2 x)^{7/2} = (1-\cos^2 x)^{7/2}$

Find $\int \sin^5 x \cos^2 x \, dx$. $= \int \sin^4 x \cdot \sin x \cdot \cos^2 x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$(a-b)^2 = a^2 - 2ab + b^2$

$(1-u^2)^2 = 1 - 2u^2 + u^4$

$$= -\int (\sin^2 x)^2 u^2 \, du$$

$$= -\int (1-u^2)^2 \cdot u^2 \, du$$

$$= -\int (1 - 2u^2 + u^4) u^2 \, du$$

$$= \int (-u^2 + 2u^4 - u^6) \, du$$

$$= -\frac{u^3}{3} + \frac{2}{5}u^5 - \frac{u^7}{7} + \text{ALNASSR}$$

$$= -\frac{\cos^3 x}{3} + \frac{2}{5}\cos^5 x - \frac{\cos^7 x}{7} + \text{ALNASSR}$$

$$\int \cos^5 x \sin^2 x \, dx = \int \cos^4 x \cdot \cos x \cdot \sin^2 x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int (1-u^2)^2 u^2 \, du$$

$$= \int (1 - 2u^2 + u^4) u^2 \, du$$

$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7} + \text{Taliscа}$$

$$= \frac{\sin^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7} + \text{Talisca}$$

$\int \sin^5 x \cos^2 x \, dx = ?$

Calculate

The best method is save one sin factor ....

5.5
6.1
6.2
6.3
+ 7.1 and 7.2] midterm 1 ✓

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$ ($\sin x$ marked $-du$)

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

$a\ln x = \ln x^a$

$$= -\int \frac{du}{u} = -\ln|u| + \text{Talisca}$$

$$= -\ln|\cos x| + \text{Talisca}$$

$$= \ln|\cos x|^{-1} + \text{Talisca}$$

$$= \ln\frac{1}{|\cos x|} + \text{Talisca} = \ln|\sec x| + \text{Talisca}$$

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$u = \sec x + \tan x$
$du = (\sec x\tan x + \sec^2 x)\,dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \int \frac{du}{u}$$

$$= \ln|u| + \text{Talisca} = \ln|\sec x + \tan x| + \text{Talisca}$$ ✓

$$\int \sec x \tan x\,dx = \sec x + \text{Talisca}$$

$$\int \frac{1}{\cos x}\,\frac{\sin x}{\cos x}\,dx$$

$u = \cos x$
$du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

$$-\int \frac{du}{u^2} = -\int u^{-2}\,du = -\frac{u^{-1}}{-1} + \text{Talisca}$$

$$= \frac{1}{u} + \text{Talisca} = \frac{1}{\cos x} + \text{Talisca}$$

$$= \sec x + \text{Talisca}$$

$$\int \sec^2 x \tan x\,dx =$$

$u = \tan x$
$du = \sec^2 x\,dx$

$$\int u \cdot du = \frac{u^2}{2} + \text{Talisca}$$

$$= \frac{\tan^2 x}{2} + \text{Talisca}$$

$$\int \sec x \tan^2 x\,dx = \int \sec x \cdot \tan x \cdot \tan x\,dx$$

$u = \tan x$
$du = \sec^2 x\,dx$

$u = \sec x$
~~$du = \sec x \tan x\,dx$~~

$$= \int \sqrt{u^2 - 1}\,.\,du$$

$$= (7.3)$$

$\tan^2 x = \sec^2 x - 1$

$\tan x = \sqrt{\sec^2 x - 1}$

$$\int \tan^3 x\,dx = \int \tan^2 x \cdot \tan x\,dx$$ ($\tan^2 x = (\sec^2 x - 1)$)

$$= \int \tan x \cdot \sec^2 x - \int \tan x\,dx = \frac{\tan^2 x}{2} + \ln|\sec x| + \text{Talisca}$$

$$\int \sec^3 x\,dx = \int \sec^2 x \cdot \sec x\,dx$$
$$= \int (\tan^2 x + 1)\cdot \sec x\,dx$$
$$= \int \tan^2 x \cdot \sec x\,dx + \int \sec x\,dx$$
$$(\sec^2 x - 1)\cdot \sec x\,dx \qquad \text{~~+ ln|sec x + tan x|~~}$$

$$\int \sec^3 x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx + \int \sec x\,dx$$

$0 = 0!$

$0 = 0!$

$f = \sec x \quad g' = \sec^2 x$

$f' = \sec x \tan x \quad g = \tan x$

$$\int \sec^3 x\,dx =$$
$$= \sec x \tan x - \int \sec x \tan^2 x\,dx \qquad (\sec^2 x - 1)$$
$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$
$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$
$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + \text{[illegible]}$$

Strategy for Evaluating $\int \tan^m x \sec^n x\,dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x\,dx = \int \tan^m x\,(\sec^2 x)^{k-1} \sec^2 x\,dx$$
$$= \int \tan^m x\,(1 + \tan^2 x)^{k-1} \sec^2 x\,dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x\,dx = \int (\tan^2 x)^k \sec^{n-1} x\ \sec x\ \tan x\,dx$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x\ \sec x\ \tan x\,dx$$

Then substitute $u = \sec x$.

$$\int \sec^6 x \cdot \tan^5 x = \int \sec^4 x\,\sec^2 x \cdot \tan^5 x\,dx$$

$(\sec^2 x)^2$

$(1 + \tan^2 x)^2$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int (1 + u^2)^2\, u^5\,du$$

$$\int (1+u^2)^2 u^5 \, du$$

$(a+b)^2 = a^2 + 2ab + b^2$.

$$= \int (1+2u^2+u^4) u^5 \, du = \frac{u^6}{6} + 2\frac{u^8}{8} + \frac{u^{10}}{10} + \text{Talisca}$$

$$= \frac{\tan^6 x}{6} + \frac{1}{4}\tan^8 x + \frac{\tan^{10}}{10} + \text{Talisca}$$

7.2 → identity
→ simplfy
→ substitution

Next time: I finish 7.2 and start 7.3

$\int \tan^5 x \sec^7 x\, dx = \int \tan^5 x \cdot \sec^5 x \sec^2 x\, dx$

$u = \tan x$, $du = \sec^2 x\, dx$ → does not work

$u^5 (1+u^2)^{5/2}\, du$

$\sec^5 x = \sqrt{(1+\tan^2 x)^5} = (1+u^2)^{5/2}$

$u = \sec x$

$du = \sec x \tan x\, dx$

$\int \tan^5 x \sec^7 x = \int \tan^4 x \cdot \tan x \cdot \sec^6 x \cdot \sec x\, dx$

$(\tan^2 x)^2 = (\sec^2 x - 1)^2 = (u^2-1)^2$, $\sec^6 x = u^6$

$\tan^2 x = \sec^2 x - 1$

$= \int (u^2-1)^2 \cdot u^6 \cdot du = \int (u^4 - 2u^2 + 1) u^6\, du$

$(a-b)^2 = a^2 - 2ab + b^2$

$= \frac{u^{11}}{11} - \frac{2}{9} u^9 + \frac{u^7}{7} + \text{Talisca}$

$= \frac{\sec^{11} x}{11} - \frac{2}{9}\sec^9 x + \frac{\sec^7 x}{7} + CR7$

7.2: → Substitution; → identities → Power Rule; → simplify

2 To evaluate the integrals (a) $\int \sin mx \cos nx\, dx$, (b) $\int \sin mx \sin nx\, dx$, or (c) $\int \cos mx \cos nx\, dx$, use the corresponding identity:

(a) $\sin A \cdot \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$\int \sin 9x \cdot \cos 2x \neq \int \sin 9x\, dx \cdot \int \cos 2x\, dx$

$\frac{1}{2}\left[\int \sin 7x + \int \sin 11x\right] dx$

$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x - \frac{1}{11}\cos 11x\right] + CR7$

$\int \sin 7x\, dx = \frac{1}{7}\int \sin u \cdot du = \frac{1}{7} - \cos u$

$u = 7x$, $du = 7dx$, $\frac{du}{7} = dx$

in midterm 1: 5.5, 6.1, 6.2, 6.3 + 7.1 + 7.2

Monday: 7.3 + Review ✓

Wednesday: 9-11 : S1.36 to answer your question. 12-1: Exam

13 Correct / 15 questions

$13 \times 1.5 = 19.5$ → A → m1

14 Correct (or 15) → m1: 20/20, Q1: 10/10

$19.5/2 = 9.75$ → Q1 A

7.3

$\int 2x\sqrt{1-x^2}\,dx$

$u = 1-x^2$

$du = -2x\,dx$

$-du = 2x\,dx$

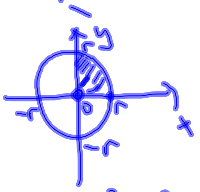
$-\int \sqrt{u}\,du = -\int u^{1/2}\,du$

$= -\frac{2}{3}u^{3/2} + \text{CR7}$

$= -\frac{2}{3}(1-x^2)^{3/2} + \text{Talaca}$

$\int \sqrt{1-x^2}\,dx$. (7.3)

6.1 Problem: Set up the integral to calculate the area of the circle $x^2+y^2=r^2$



$x^2+y^2=r^2$

$y^2=r^2-x^2$

$y=\sqrt{r^2-x^2}$

$\text{Area} = 4\cdot\int_0^r y\,dx$

$= 4\int_0^r \sqrt{r^2-x^2}\,dx$ (7.3)

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$ ✓ $\sqrt{r^2-x^2}$	$x = a\sin\theta$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ ✓ $x = r\sin\theta$	$1-\sin^2\theta = \cos^2\theta$ ✓
$\sqrt{a^2+x^2}$	$x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

↓

Simplify

↓

Solve the integral

$x = r\sin\theta$ $0 = r\sin\theta$, $\sin\theta = 0$, $\theta = 0$

$dx = r\cos\theta\,d\theta$ $r = r\sin\theta$, $\sin\theta = 1$, $\theta = \pi/2$

$x^2 = r^2\sin^2\theta$

$\text{Area} = 4\int_0^r \sqrt{r^2-x^2}\,dx$

$= 4\int_0^{\pi/2} \sqrt{r^2-r^2\sin^2\theta}\; r\cos\theta\,d\theta$

$r^2(1-\sin^2\theta)$, $\cos^2\theta$, $r\cos\theta$

$= 4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta = 4r^2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$

$= 4r^2\,\frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= 2r^2\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)-\left(0+\frac{1}{2}\sin 0\right)\right]$

$= 2r^2\cdot\frac{\pi}{2}$

$= \pi r^2$ ✓

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution 1:

$x = a\sec\theta$

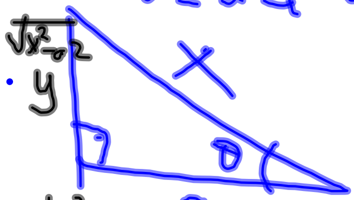
$dx = a\sec\theta \cdot \tan\theta \, d\theta$

$x^2 = a^2\sec^2\theta$

$x = a \cdot \sec\theta = a \cdot \frac{1}{\cos\theta}$

$\frac{x}{1} = \frac{a}{\cos\theta}$

$x\cos\theta = a \Rightarrow \cos\theta = \frac{a}{x}$



$y^2 + a^2 = x^2$

$y = \sqrt{x^2 - a^2}$

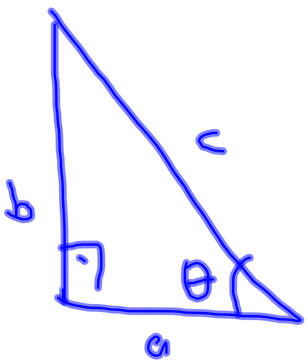
$$\int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$$

$a^2(\sec^2\theta - 1) = a^2\tan^2\theta$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta}$$

$$= \int \sec\theta\, d\theta \overset{7.2}{=} \ln|\sec\theta + \tan\theta| + CR7$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + CR7$$



$c^2 = a^2 + b^2$

$\cos\theta = \frac{a}{c}$ $\quad \tan\theta = \frac{b}{a}$

$\sin\theta = \frac{b}{c}$ $\quad \cot\theta = \frac{a}{b}$

$\int \frac{dx}{\sqrt{x^2 - 3}}$ is given. $\quad x^2 - 3 = x^2 - (\sqrt{3})^2$ ✓

- Calculate ✓
- which subst. is the best : $x = \sqrt{3}\sec\theta$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$, $x^2 = 9\sin^2\theta$

$\sqrt{9(1-\sin^2\theta)} = \sqrt{3^2\cos^2\theta}$

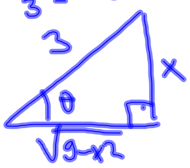
$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx. = \int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} 3\cos\theta\, d\theta$$

$$= \int \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + CR7$$

$x = 3\sin\theta \Rightarrow \frac{x}{3} = \sin\theta$, $\arcsin\frac{x}{3} = \theta$

$$= -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + CR7$$



Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$ ✗	$x = a\sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$ ($\sqrt{x^2+2^2}$)	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ($x = 2\tan\theta$)	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$ ✗	$x = a\sec\theta,\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $x^2 = 4\tan^2\theta$

$\sqrt{4(\tan^2\theta + 1)} = \sqrt{2^2\sec^2\theta}$

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx. = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta + 4}}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta\, d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$u = \sin\theta$, $du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\int u^{-2}\, du$$

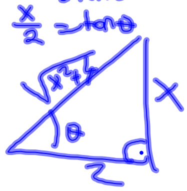
$x \to \theta \to u$

$$= -\frac{1}{4}u^{-1} + \text{AL NASSR}$$

$$= -\frac{1}{4u} + \text{AL NASSR}$$

$$= -\frac{1}{4\sin\theta} + \text{AL NASSR}$$

$x = 2\tan\theta$, $\frac{x}{2} = \tan\theta$



$$= -\frac{1}{4 \cdot \frac{x}{\sqrt{x^2+4}}} + \text{AL NASSR}$$

$$= -\frac{\sqrt{x^2+4}}{4x} + \text{AL NASSR}$$

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx.$$

- Calculate
- Which subst. is most helpfull
  $x = 2\tan\theta$
- Which identity is the best to simplify
  $1 + \tan^2\theta = \sec^2\theta$

Review for Midterm1 Test
(5.5, 6.1, 6.2, 6.3, 7.1, 7.2)

5.5

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u} \cdot 2u\,du$

$u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$, $dx = 2\sqrt{x}\,du = 2u\,du$

$= 2\int \sin u\,du = -2\cos u + \text{Tolis}(a)$

$= -2\cdot\cos\sqrt{x} + \text{Tolis}(a)$

$\int \sin x \sin(\cos x)\,dx =$

$u = \cos x$, $du = -\sin x\,dx$, $-du = \sin x\,dx$

$\int \sin u \cdot -du = -\int \sin u\,du$

$= \cos u + \text{DURAN}$

$= \cos(\cos x) + \text{DURAN}$

$h(x) = \int_1^{e^x} \ln t\,dt \Rightarrow h'(x) = ?$

$S(x) = \int_1^{x} \ln t\,dt$

$S'(x) = \ln x$

$S'(e^x) = \ln e^x = x$

$h(x) = S(e^x)$ (inner)

$h'(x) = e^x \cdot S'(e^x)$

$= e^x \cdot x$

$h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz$

$S(x) = \int_1^{x} \frac{z^2}{z^4+1}\,dz$

$S'(x) = \frac{x^2}{x^4+1}$

$S'(\sqrt{x}) = \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1}$

$h(x) = S(\sqrt{x})$

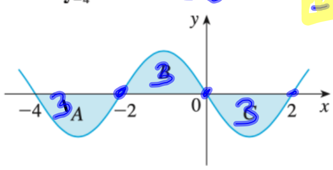
$h' = \frac{1}{2\sqrt{x}} S'(\sqrt{x})$

$= \frac{1}{2\sqrt{x}} \cdot \frac{x}{x^2+1}$

Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 3. Find the value of

$\int_{-4}^{2} [f(x) + 2x + 5]\,dx = \int_{-4}^{2} f(x)\,dx + \int_{-4}^{2} (2x+5)\,dx$

$\int_{-4}^{2} (2x+5)\,dx = 18$



$\int_{-4}^{-2} f\,dx + \int_{-2}^{0} f\,dx + \int_{0}^{2} f\,dx = -3 + 3 + -3$

$-3 + 18 = 15$

Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$$

$F = 3$, $x < 3$; $F = x$, $x > 3$; 0, 3, 5

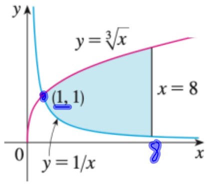
$$\int_0^5 F\,dx = \int_0^3 \underbrace{F}_{3}\,dx + \int_3^5 \underbrace{F}_{x}\,dx$$

$$= 3x\Big|_0^3 + \frac{x^2}{2}\Big|_3^5 = 9 + \frac{25}{2} - \frac{9}{2}$$

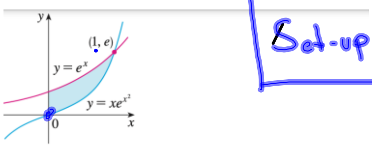
$$= 9 + 8 = 17 \checkmark$$

6.1

Set up: $\int_1^8 \left(\sqrt[3]{x} - \frac{1}{x}\right) dx \checkmark$



$$\int_1^8 \sqrt[3]{x} - \frac{1}{x}\,dx$$
$$= \int_1^8 x^{1/3} - \frac{1}{x}\,dx$$
$$= \left[\frac{3}{4}x^{4/3} - \ln x\right]_1^8$$
$$= \left[\frac{3}{4}x\sqrt[3]{x} - \ln x\right]_1^8$$
$$= \left[\frac{3}{4}\cdot 8\sqrt[3]{8} - \ln 8\right] - \left[\frac{3}{4}\cdot 1\sqrt[3]{1} - \ln 1\right]$$
$$= \frac{48}{4} - \ln 8 - \frac{3}{4}$$



Set-up: $\int_0^1 \left(e^x - xe^{x^2}\right) dx$

$$\int_0^1 e^x - xe^{x^2}\,\mathrm{d}x \qquad (1)$$

Now lets separe it into 2 integrals and solve them:

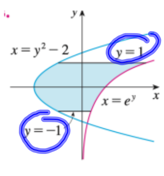
$$\int_0^1 e^x\,\mathrm{d}x - \int_0^1 xe^{x^2}\,\mathrm{d}x$$
$$\int_0^1 e^x\,\mathrm{d}x = e^x\Big|_0^1 = e - 1$$

For the other integral we will use the following substitution:

$$u = x^2\,\mathrm{d}x \implies \mathrm{d}u = 2x\,\mathrm{d}x \implies \frac{\mathrm{d}u}{2x} = \mathrm{d}x$$

Now we have:

$$\int_0^1 xe^u\frac{1}{2x}\,\mathrm{d}u$$
$$\frac{1}{2}\int_0^1 e^u = \frac{1}{2}e^u\Big|_0^1 = \frac{e}{2} - \frac{1}{2}$$



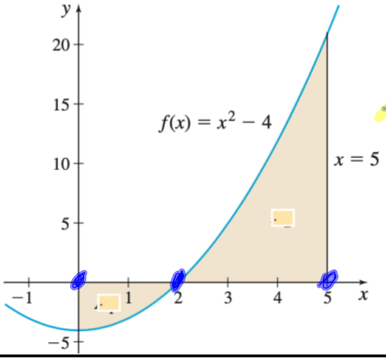
Set-up

$$\int_{-1}^{1}\left[e^y - (y^2 - 2)\right] dy$$

$$\int_{-1}^{1}\left[e^y - y^2 + 2\right] dy$$

$$A = \int_{-1}^{1}\left[e^y - (y^2 - 2)\right] dy$$
$$= \int_{-1}^{1}\left(e^y - y^2 + 2\right) dy$$
$$= \int_{-1}^{1} e^y dy - \int_{-1}^{1} y^2 dy + 2\int_{-1}^{1} dy$$

$$A = \int_{-1}^{1}\left[e^y - (y^2 - 2)\right] dy$$
$$= \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1}$$
$$= \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right)$$
$$= e - \frac{1}{e} + \frac{10}{3}$$



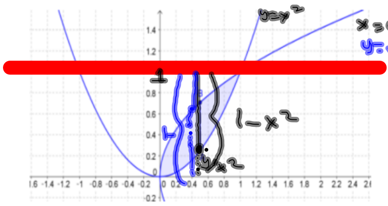
Set-up

$$-\int_0^2 (x^2-4)\,dx + \int_2^5 (x^2-4)\,dx$$

$$\int_0^2 (4-x^2)\,dx + \int_2^5 (x^2-4)\,dx$$

6.2

$y = x^2$, $x = y^2$; about $y = 1$



Set-up

$$\int_0^1 \left[(1-x^2)^2 - (1-\sqrt{x})^2\right] dx$$

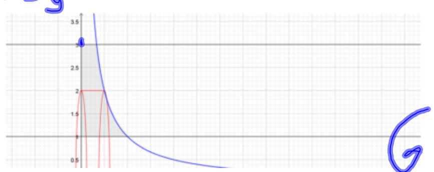
exercise: Shell method

6.3

$xy = 1$, $x = 0$, $y = 1$, $y = 3$

about x axis

$x = \frac{1}{y}$



Set-up: $\int_1^3 2\pi y \,\text{height}\, dy$

$$\int_1^3 2\pi y \frac{1}{y}\, dy$$

$$2\pi \int_1^3 1\, dy$$

$$2\pi\, y\Big|_1^3 = 4\pi$$

exercise: Solve by disk

$\int (\ln x)^2\,dx = fg - \int g f'\,dx$

$f = (\ln x)^2 \quad g' = 1$

$f' = \frac{1}{x}\cdot 2\ln x, \quad g = x$

$= x\ln^2 x - \int x \cdot \frac{1}{x} 2\ln x\,dx$

$= x\ln^2 x - 2\int \ln x\,dx$ → in the class notes, integration by parts

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

$\frac{x}{(1+2x)^2} = g'$

$g =$

$f = xe^{2x}$ $\quad g' = \frac{1}{(1+2x)^2}$

$f' = e^{2x} + x \cdot 2e^{2x}$

$g =$

$fg - \int g f' dx$

Integration by parts
$u = xe^{2x}$ $\quad du = e^{2x} + 2xe^{2x} = e^{2x}(1+2x)$
$dv = \frac{1}{(1+2x)^2}$ $\quad v = \frac{-1}{2(1+2x)}$

$$I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

$$= xe^{2x} \cdot \frac{-1}{2(1+2x)} - \int e^{2x}(1+2x) \cdot \frac{-1}{2(1+2x)}\,dx$$

$$= -\frac{xe^{2x}}{2(1+2x)} - \int \frac{-1}{2} e^{2x}\,dx$$

$$= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}\,dx$$

$$= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C$$

$$= -\frac{xe^{2x}}{2(1+2x)} + \frac{e^{2x}}{4} + C$$

$$= -\frac{2xe^{2x}}{4(1+2x)} + \frac{(1+2x)e^{2x}}{4(1+2x)} + C$$

$u = 1+2x$

$du = 2dx$

$\frac{du}{2} = dx$

$$\int \frac{dx}{(1+2x)^2} = \frac{1}{2}\int \frac{du}{u^2}$$

...

7.2

Please check the File in the Moodle.

(Time is over)

Wednesday: 9-11 S1.36 ✓

12-1: Exam.

7-4

Rational $f(x) = \frac{P(x)}{Q(x)}$ ← polynomials.

$\frac{1}{x}, \frac{1}{x+1}, \frac{2x}{x^2+1}, \frac{1}{x^2+1}, \frac{x^4}{x^2+1}$.

$\int \frac{1}{x}\,dx = \ln|x| + C$ ✓

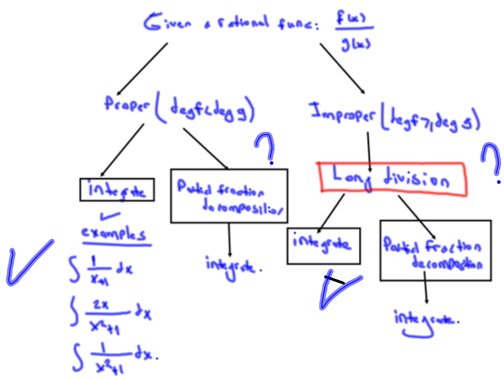
$\int \frac{1}{x+1}\,dx = \int \frac{du}{u} = \ln|u| + C$
$= \ln|x+1| + C$

$u = x+1$
$du = dx$

$\int \frac{2x}{x^2+1}\,dx = \int \frac{du}{u} = \ln|u| + C$
$= \ln(x^2+1) + C$

$u = x^2+1$
$du = 2x\,dx$

$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$



$\int \frac{x^3 + x}{x - 1}\,dx.$ (371)

divisor: $x - 1$

quotient: $x^2 + x + 2$

$x \cdot ? = x^3$
$x \cdot ? = x^2$
$x \cdot ? = 2x$

$x^3 + x$
$-\,x^3 - x^2$
$x^2 + x$
$-\,x^2 - x$
$2x$
$-\,2x - 2$
2 Remainder

deg(Remainder) < deg(divisor)

Rational = quotient + $\frac{\text{Remainder}}{\text{divisor}}$

$\int \frac{x^3+x}{x-1}\,dx = \int \left[x^2 + x + 2 + \frac{2}{x-1}\right] dx$

$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$

$\int \frac{2}{x-1}\,dx = 2\int \frac{1}{x-1}\,dx$

$u = x-1$ $du = dx$ $2\int \frac{du}{u} = 2\ln|u|$

$\int \frac{4x^3+6x^2-10x+4}{2x-1}\,dx$. (371)

$4x^3+6x^2-10x+4 \;|\; 2x-1$ divisor

quotient: $2x^2+4x-3$

$-(4x^3-2x^2)$

$8x^2-10x+4$

$-(8x^2-4x)$

$-6x+4$

$-(-6x+3)$

1

$2x \cdot ? = 4x^3$

$2x \cdot ? = 8x^2$

$2x \cdot ? = -6x$

$\int \frac{4x^3+6x^2-10x+4}{2x-1}\,dx = \int \left[2x^2+4x-3+\frac{1}{2x-1}\right]dx$

$= 2\frac{x^3}{3}+2x^2-3x+\frac{1}{2}\ln|2x-1|+C$

$\int \frac{1}{2x-1}dx$. $u=2x-1$ $du=2dx$ $\frac{du}{2}=dx$

$\int \frac{du/2}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

- $\frac{4x^3+6x^2-10x+4}{2x-1} = f(x)$ $\quad \int f(x)dx = ?$

Which one is TRUE?

a) f is proper F

b) degree of numerator = 1 F

c) $f = 2x^2+4x-3+\frac{1}{2x-1}$ ✓

d) degree of denominator = 3 F

2. $\frac{4x^3+6x^2-10x+4}{2x-1} = P(x) + \frac{R(x)}{Q(x)} \Rightarrow P(2) = ?$

$P(x) = 2x^2+4x-3$

$P(2) = 2\cdot 2^2+4\cdot 2-3 = 8+8-3 = 13$.

Partial Fraction decomposition

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Case I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another).

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

EXAMPLE 3 Find $\int \frac{dx}{x^2 - 16}$, where $a \neq 0$.

$$\int \frac{dx}{x^2-16} = \int \boxed{\frac{1}{x^2-16}}\, dx$$

$0 < 2 \Rightarrow$ proper (no need for long division)

$$\frac{1}{\underbrace{x^2-16}_{\substack{x^2-4^2 \\ (x-4)(x+4)}}} = \frac{A}{x-4} + \frac{B}{x+4}$$

$$(x^2-16)\cdot \frac{1}{x^2-16} = \left(\frac{A}{x-4} + \frac{B}{x+4}\right)\overbrace{(x^2-16)}^{(x-4)(x+4)}$$

$$1 = A(x+4) + B(x-4)$$

$x=4$: $1 = 8A + B\cdot 0$

$A = \frac{1}{8}$.

$x=-4$: $1 = A\cdot 0 + B\cdot -8 \Rightarrow B = -\frac{1}{8}$

$$\int \frac{1}{x^2-16}\,dx = \int \left[\frac{\frac{1}{8}=A}{x-4} + \frac{-\frac{1}{8}=B}{x+4}\right] dx$$

$\int \frac{1}{x-4}\,dx \cdot u = x-4$, $du = dx$

$\int \frac{du}{u} = \ln|u| = \ln|x-4|$

$$= \frac{1}{8}\ln|x-4| - \frac{1}{8}\ln|x+4| + C \in \mathbb{R}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx. \quad (2 < 3) \text{ no need for long division}$$

$2x^3+3x^2-2x$

$x(2x^2+3x-2)$

$2x \quad x \qquad 2 \quad -1 \qquad 1 \quad -2$

$x(2x-1)(x+2)$

$$~~x(2x-1)(x+2)~~\left(\frac{x^2+2x-1}{~~x(2x-1)(x+2)~~}\right) = \left(\frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}\right)x(2x-1)(x+2)$$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$x=0$: $-1 = A\cdot -1\cdot 2$

$-1 = -2A$

$A = 1/2$

$x=-2$:

$4-4-1 = C\cdot -2\cdot -5$

$-1 = 10C \Rightarrow C = -\frac{1}{10}$

$x=\frac{1}{2}$: $\frac{1}{4} + 1 - 1 = B\cdot\frac{1}{2}\cdot\frac{5}{2}$

$\frac{1}{4} = \frac{5B}{4} \Rightarrow B = \frac{1}{5}$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx. = \int\left(\frac{\frac{1}{2}}{x} + \frac{\frac{1}{5}}{2x-1} + \frac{-\frac{1}{10}}{x+2}\right)dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C \in \mathbb{R}$$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Case II $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

7 $$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Set-up partial fraction decomposition

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

$x \cdot x$ $(x-1)(x-1)(x-1)$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx.$$

$x^3 \cdot ? = x^4$

$x^3 \cdot ? = x^3$

$$\begin{array}{l} x^4 - 2x^2 + 4x + 1 \;|\; x^3 - x^2 - x + 1 \\ -\;(x^4 - x^3 - x^2 + x) \qquad x + 1 \\ \hline x^3 - x^2 + 3x + 1 \\ -\;(x^3 - x^2 - x + 1) \\ \hline 4x \end{array}$$

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

$(x-1)^2(x+1)$: given

$$\frac{4x}{(x-1)^2(x+1)} = \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\right)(x-1)^2(x+1)$$

$x=1$ $x=1$ $x=-1$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$x=1$: $4 = 2B \Rightarrow B = 2$

$x=-1$: $-4 = 4C \Rightarrow C = -1$

$x=0$: $0 = -A + 2 - 1 \Rightarrow 0 = -A + 1 \Rightarrow A = 1$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$\int \frac{2}{(x-1)^2}\,dx$. $u = x-1$, $du = dx$

$$2\int \frac{du}{u^2} = 2\int u^{-2}\,du$$

$$= \frac{2}{-1}u^{-1} = -\frac{2}{u} = -\frac{2}{(x-1)}$$

In the exam

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}.$$

① Perform Long division!

② Setup partial fraction $\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$ax^2+bx+c \rightarrow b^2-4ac<0$

Case III $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

9 $$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

ax^2+bx+c
$\Delta = b^2-4ac$

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

Case 1

x^2+1: $a=1, b=0, c=1$
$\Delta = 0^2-4\cdot1\cdot1 = -4<0$
x^2+4: $a=1, b=0, c=4$
$\Delta = 0^2-4\cdot1\cdot4 = -16<0$

$$\text{Evaluate } \int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx.$$

$x(x^2+4)$

$$\frac{2x^2-x+4}{x(x^2+4)} = \left(\frac{A}{x} + \frac{Bx+C}{(x^2+4)}\right) x(x^2+4)$$

Case 1, Case 3; $x=0$; $x^2+4=0 \Rightarrow x^2=-4$!

$2x^2-x+4 = A(x^2+4) + (Bx+C)\cdot x$

$-1(-B+C)$
$B-C$

$x=0$: $4 = 4A \Rightarrow A = 1.$

$x=1$: $5 = 5 + B+C \Rightarrow B+C=0$

$x=-1$: $7 = 5 + B - C \Rightarrow + \; B-C=2$

$2B=2 \Rightarrow B=1$
$C=-1$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx$$

$$= \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$\int \frac{x}{x^2+4}dx$ $\quad u=x^2+4$, $du=2x\,dx$

$\int \frac{\frac{du}{2}}{u} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|$

Case IV $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (9), the sum

11 $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

Case 1 Case 1 Case 3 Case 4

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+1)^1} + \frac{Kx+L}{(x^2+1)^2} + \frac{Mx+N}{(x^2+1)^3}$$

$1x^2 + 1x + 1$:
$ax^2 + bx + c$:
$a = 1$ $b = 1$ $c = 1$
$\Delta = b^2 - 4ac$
$1^2 - 4 \cdot 1 \cdot 1$
$-3 < 0$

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} \, dx.$$

Case 1 Case 4

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

(ex) which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$ and $E = 0$.

$$\int \left[\frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2} \right] dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \arctan x - \frac{1}{2(x^2+1)} + C$$

$u = x^2+1$ $du = 2x\,dx$

$$\frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du$$

$$\frac{1}{2}\frac{u^{-1}}{-1} = -\frac{1}{2u}$$

Rationalizing Substitutions

$$\int \frac{\sqrt{x+4}}{x}\,dx.$$

$u^2 = x+4 \Rightarrow (x = u^2-4)$

$2u\,du = dx$

$$\int \frac{\sqrt{u^2}\cdot 2u\,du}{u^2-4} = 2\int \boxed{\frac{u^2}{u^2-4}}\,du \quad \longrightarrow \text{rational function}$$

$u^2 \cdot ? = u^2$

Long division: $u^2 \div (u^2-4) = 1$, $u^2 - (u^2-4) = 4$

$$\frac{u^2}{u^2-4} = 1 + \boxed{\frac{4}{u^2-4}}$$

$$\frac{4}{(u-2)(u+2)} = \left(\frac{A}{u-2} + \frac{B}{u+2}\right)(u-2)(u+2) \quad \text{(case 1)}$$

$u=2$, $u=-2$

$4 = A(u+2) + B(u-2)$

$u=2$: $4 = 4A \Rightarrow A = 1$

$u=-2$: $4 = -4B \Rightarrow B = -1$

$$\int \frac{u^2}{u^2-4}\,du = \int \left[1 + \frac{1}{u-2} + \frac{-1}{u+2}\right] du$$

$$= u + \ln|u-2| - \ln|u+2| + C \in \mathbb{R}$$

$u = \sqrt{x+4}$

$$2\left[\sqrt{x+4} + \ln\left|\sqrt{x+4}-2\right| - \ln\left|\sqrt{x+4}+2\right|\right] + C \in \mathbb{R}$$

non-rational
↓
Rational
↓
improper
↓
Long Division → Partial Fraction → integration

7.8

Improper integrals

↓

Improper refers different concepts in 7.4 and 7.8

Before we dive into 7.8, let me cover L'hopital's Rule (we need it in 7.8)

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$

if the limit on the right side exists (or is ∞ or $-\infty$).

EXAMPLE 1 Find $\lim_{x\to1}\frac{\ln x}{x-1}$. $= \frac{0}{0}$.

$$\lim_{x\to1}\frac{\ln x}{x-1} = \lim_{x\to1}\frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1 \checkmark$$

EXAMPLE 2 Calculate $\lim_{x\to\infty}\frac{e^x}{x^2}$. $= \frac{e^\infty}{\infty^2} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} \; \frac{\left(\frac{\infty}{\infty}\right)}{} \; \lim_{x\to\infty}\frac{e^x}{2} = \frac{\infty}{2} = \infty.$$

Example 1
Evaluate the limit $\lim_{x\to3}\frac{x^2+x-12}{x^2-9}$ using $= \frac{0}{0}$

Calculus 1 : $\lim_{x\to3}\frac{(x+4)(x-3)}{(x-3)(x+3)} = \frac{7}{6} \checkmark$

L'hopital's rule : $\lim_{x\to3}\frac{2x+1}{2x} = \frac{7}{6} \checkmark$

EXAMPLE 5 Find $\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}$. $= \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = \frac{0}{2} = 0!$

Apply L'hopital's rule (blindly!)

$$\lim_{x\to\pi^-}\frac{\cos x}{\sin x} = \frac{\cos\pi}{\sin\pi} = \frac{-1}{0} = -\infty \text{ (wrong)}$$

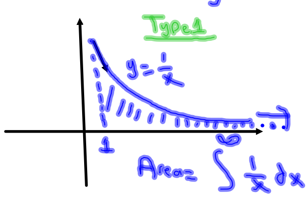
Example 4
Evaluate the limit $\lim_{x\to1}\frac{\sqrt{2-x}-x}{x-1}$ using L'Hopital's Rule. $= \frac{0}{0}$

$\sqrt{u} = \frac{u'}{2\sqrt{u}}$, $u = 2-x$

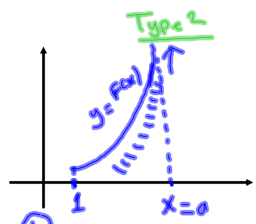
L'hopital's rule : $\lim_{x\to1}\frac{\frac{-1}{2\sqrt{2-x}}-1}{1} = \frac{-\frac{1}{2}-1}{1} = -3/2$.

7.8

why 7.8



Area $= \int_1^\infty \frac{1}{x}\,dx$



$\int_0^b f(x)\,dx$ = Are a

Type 1

$\int_{-\infty}^{\infty} f\,dx, \int_{-\infty}^{0} f\,dx, \int_{a}^{\infty} f\,dx.$

Type 2

$\int_a^b f(x)\,dx$ ← not continous (at b) and/or not continuous (at a)

---

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

Calculate:

$\int_1^\infty (1/x)\,dx$ (Type 1)

$$\lim_{t\to\infty}\left[\int_1^t \frac{1}{x}\,dx\right] = \lim_{t\to\infty}\left[\ln x \Big|_1^t\right]$$

$$= \lim_{t\to\infty}\left[\ln t - \ln 1\right] \quad (\ln 1 = 0)$$

$$= \lim_{t\to\infty} \ln t = \ln\infty = \infty$$

divergent

mcq: $\int_1^\infty \frac{1}{x}\,dx$ is given. Which one is TRUE

a) Type 2
b) Convergent and equal to 1
c) Convergent and equal to 0
(d) divergent

2 $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

$\int_1^\infty \frac{1}{x^p}\,dx$ → $p>1$ → convergent $= \frac{1}{p-1}$
→ $p \leq 1$ → divergent.

$$\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$$

$p = 3 > 1$

$\frac{\text{number}}{\pm\infty} = 0$

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t x^{-3}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{x^{-2}}{-2}\Big|_1^t\right]$$

$$= \lim_{t\to\infty}\left[\frac{t^{-2}}{-2} + \left(+\frac{1^{-2}}{-2}\right)\right] = \frac{\infty^{-2}}{-2} + \frac{1}{2} = \frac{1}{-2\cdot\infty^2} + \frac{1}{2}$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

$$\int_1^\infty \frac{1}{x^{2/3}}\,dx = \text{divergent}$$

$$\left(\frac{2}{3} \leq 1\right)$$

$$\int_1^\infty \frac{1}{x^{7/6}}\,dx = \text{convergent} = \frac{1}{p-1}$$

$$= \frac{1}{\frac{7}{6}-1} = 6 \checkmark$$

$$\int_1^\infty \frac{1}{x^p}dx \quad p>1 \to \text{convergent and } \frac{1}{p-1}, \quad p\le 1 \to \text{divergent}$$

example. $\int_1^\infty \frac{1}{x^3}dx \overset{p=3>1}{=} \frac{1}{3-1} = \frac{1}{2}$ ✓

---

example $\int_2^\infty \frac{1}{x^3}dx$ $\underline{\underline{\text{common mistake}}}$ $\frac{1}{3-1} = \frac{1}{2}$ WRONG

① $\lim_{t\to\infty}\left[\int_2^t \frac{1}{x^3}dx\right] = \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big|_2^t$ (x^{-3}; $-\frac{1}{2x^2}$)

$= \lim_{t\to\infty}\left[-\frac{1}{2t^2} + \frac{1}{2\cdot 2^2}\right] = 0 + \frac{1}{8} = \frac{1}{8}$ ✓

② $\int_1^\infty x^{-3}dx = \int_1^2 x^{-3}dx + \int_2^\infty x^{-3}dx$ (1 — 2 — ∞)

$\frac{1}{2} = \frac{x^{-2}}{-2}\Big|_1^2 + ?$ ($-\frac{1}{2x^2}$) $\Rightarrow \frac{1}{2} = -\frac{1}{8} + \frac{1}{2} + ?$

$\frac{1}{8} = ? = \int_2^\infty x^{-3}dx$

$\int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty}\int_t^0 xe^x dx$ $f=x$ $g'=e^x$, $f'=1$ $g=e^x$

$\lim_{t\to-\infty}\left[xe^x\Big|_t^0 - \int_t^0 e^x dx\right]$ ($e^x\Big|_t^0$)

$\lim_{t\to-\infty}\left[(0-te^t)-(1-e^t)\right]$ ($-te^t-1+e^t$)

$\lim_{t\to-\infty} -te^t + \lim_{t\to-\infty} -1 + \lim_{t\to-\infty} e^t$

$\infty\cdot 0$ $\quad -1 + 0$

$e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

$-te^t = \frac{-t}{e^{-t}} \to \frac{\infty}{\infty}$

$\lim_{t\to-\infty}\frac{-t}{e^{-t}} = \frac{\infty}{\infty} \overset{\text{L'hopital's rule}}{=} \lim_{t\to-\infty}\frac{-1}{-1\cdot e^{-t}} = \frac{1}{\infty} = 0$

$0 + -1 + 0 = -1$ ✓

---

we are given $\int_{-\infty}^0 xe^x dx$. which one is True:

a) Type 2 (False) b) divergent (False) c) it is 0 d) it is -1.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$-\infty \quad 0 \quad \infty$

ex: $\pi/2$ ✓

$$\lim_{t\to+\infty} \int_0^t \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^t$$

$$\lim_{t\to+\infty} (\arctan t - \arctan 0)$$

$$\lim_{t\to+\infty} \arctan t = \arctan\infty = \frac{\pi}{2}.$$

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi.$$ ✓

---

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

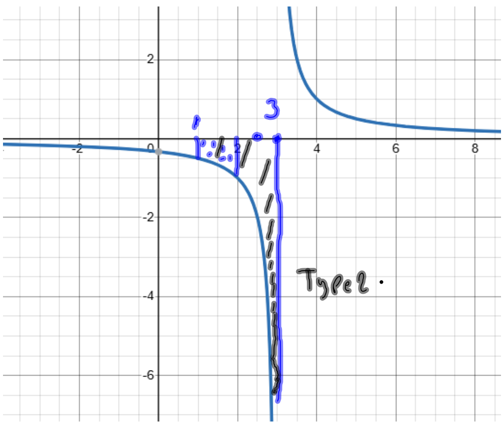
$x = b$

Calculate $\int_1^2 \frac{1}{x-3}\,dx$ (calcl integral) $1 \quad 2 \quad \cdots 3$

$$= \ln|x-3| \Big|_1^2 = \ln|-1| - \ln|-2|$$

$$= \ln 1 - \ln 2 = -\ln 2.$$

ex. $\int_1^3 \frac{1}{x-3}\,dx$ $\quad 1 \quad 3$



Calculate $\int_1^3 \frac{1}{x-3}\,dx = \lim_{t\to 3^-} \int_1^t \frac{1}{x-3}\,dx$

$= \lim_{t\to 3^-} \left[\ln|x-3| \Big|_1^t \right]$

$= \lim_{t\to 3^-} \left\{ \ln|t-3| - \ln|-2| \right\}$

$= \lim_{t\to 3^-} \left\{ \ln|t-3| - \ln 2 \right\}$ $\quad \ln 0 = -\infty$

$= -\infty - \ln 2 = -\infty$

integral is divergent.

We are given $\int_0^{3} \frac{1}{x-3}\,dx$. Which one is True

a) Type 1
(b) Type 2 (circled)
c) it is $-\ln 2$
d) it is 0
e) it is 5

We are given $\int_1^2 \frac{1}{x-3}\,dx$. Which one is True

a) Type 2 ✗
b) Type 1 ✗
c) divergent ✗
(d) it is $-\ln 2$ (circled)
e) it is 0 ✗

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$$

$= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$

$= \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5$

$= \lim_{t\to 2^+} \left[2\sqrt{3} - 2\sqrt{t-2} \right]$

$= 2\sqrt{3} - 0 = 2\sqrt{3}$

$= 2\sqrt{3}.$

$\int \frac{1}{\sqrt{x-2}}\,dx$. $u = x-2$, $du = dx$

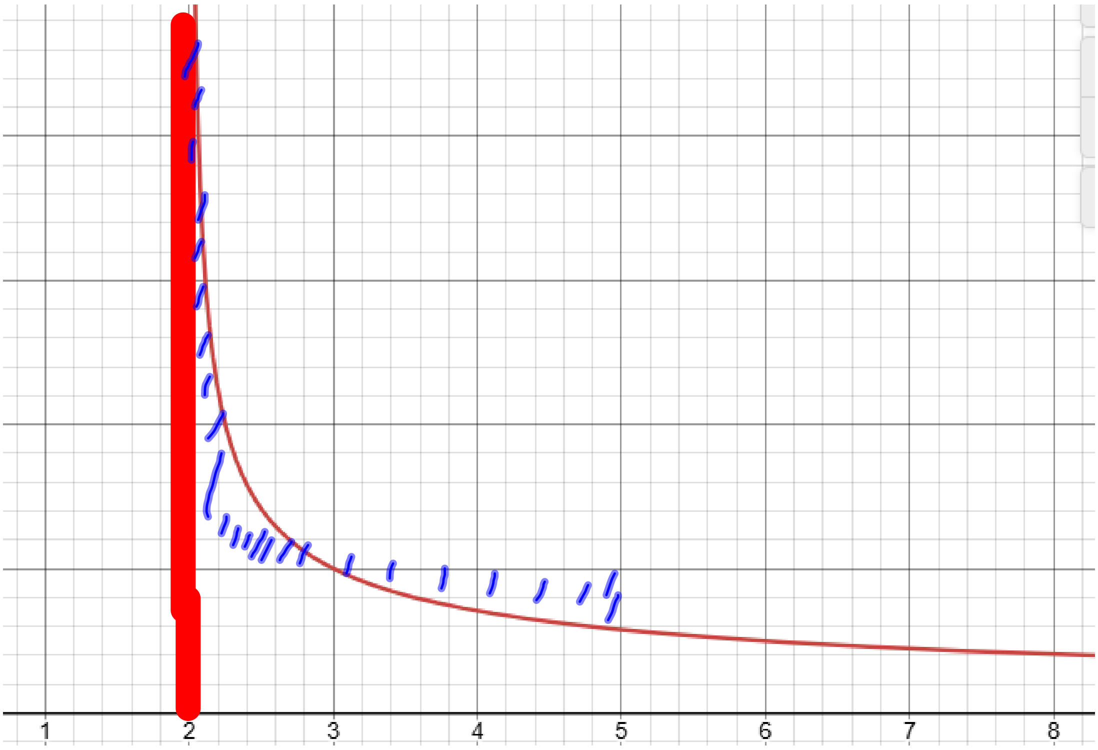
$\int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$

$\frac{u^{1/2}}{1/2} = 2u^{1/2}$

$= 2\sqrt{u} = 2\sqrt{x-2}$

$\lim_{t\to 2^+} 2\sqrt{t-2} =$

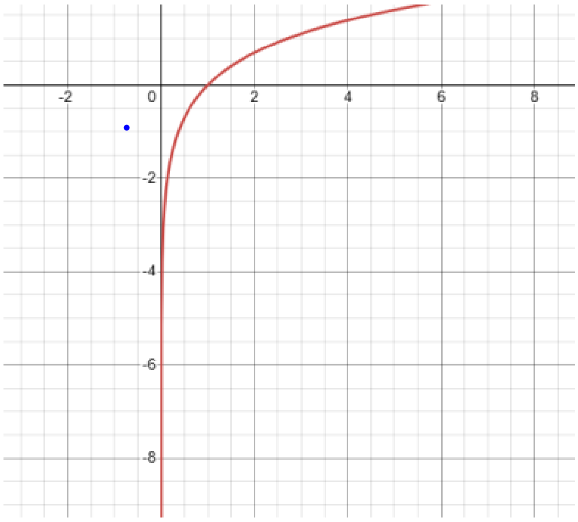
$t<2$ $\quad$ $t>2$

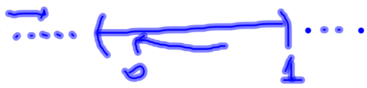


Next time : more examples and 8.1

Quiz 2 : 7.2, 7.3, 7.4, 7.8

EXAMPLE 8 $\int_0^1 \ln x\, dx$. (ln 0 is undefined)





$$\int_0^1 \ln x\, dx = \lim_{t \to 0^+} \int_t^1 \ln x\, dx$$

$f = \ln x \quad g' = 1$

$f' = \frac{1}{x} \quad g = x$

$$\lim_{t\to 0^+}\left[x\ln x \Big|_t^1 - \int_t^1 x \cdot \frac{1}{x}\, dx \right] \quad x \Big|_t^1$$

$$\lim_{t\to 0^+}\left[(1\ln 1 - t\ln t) - (1 - t) \right]$$

$$-\lim_{t\to 0^+} t\ln t + \lim_{t\to 0^+} -1 + \lim_{t\to 0^+} t$$

$-\lim_{t\to 0^+} t\ln t$ (L'H) $\Downarrow 0$ $\quad + \quad -1 \quad \quad 0$

$0 \cdot \ln 0 = 0 \cdot -\infty$

$t\ln t = \dfrac{\ln t}{\frac{1}{t}} = \dfrac{-\infty}{\infty}$

$0 \cdot -\infty$

$$\lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$\lim_{t\to 0^+} \frac{1}{t} \cdot -t^2 = \lim_{t\to 0^+} -t$$

$= 0$

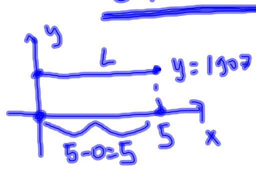
$$0 + -1 + 0 = -1$$

ex. We are given $\int_0^1 \ln x\, dx$. TRUE!

~~a) Type I~~ ~~b) divergent~~ ~~c) it is 0~~ (d) it is -1

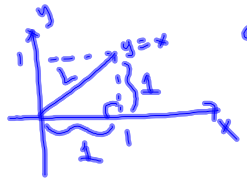
8.1 (Arclength)

(1)



a) 1 (b) 5) c) 1907
d) -1 e) 2025

(2)

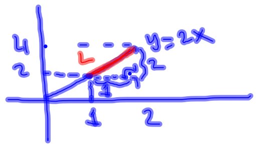


a) 0 b) 1 (c) $\sqrt{2}$) d) $\sqrt{5}$ e) 2

$L^2 = 1^2 + 1^2$

$L^2 = 2 \Rightarrow L = \sqrt{2}$

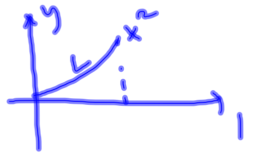
(3)



$L = \sqrt{1^2 + 2^2} = \sqrt{5}$.

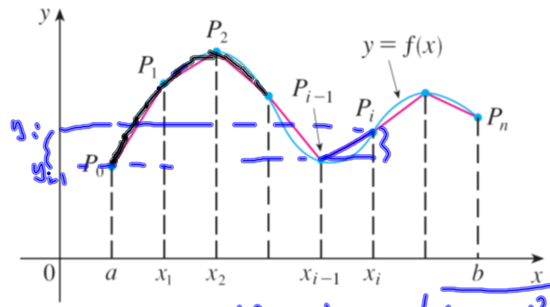
---

(4)



We need 8.1

$L = ?$



$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$

$[a,b] = [x_{i-1}, x_i]$

$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$

m 2 (Mean Value). If f is a continuous function on the closed interval $[a, b]$ which ntiable on the interval (a, b), then there's a point $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$= \sqrt{(\Delta x)^2 + f'(c)^2 \, \Delta x^2}$

$f'(c) \cdot \Delta x = \Delta y_i$

$= \Delta x \sqrt{1 + (f')^2}$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \sqrt{1 + (f')^2}$$

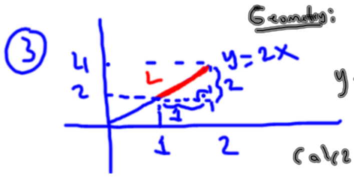
$$L = \int_a^b \sqrt{1 + (y')^2} \, dx$$

---

2 The Arc Length Formula If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

③



Geometry: $L = \sqrt{1^2+2^2} = \sqrt{5}$.

$y = 2x \Rightarrow y' = 2$

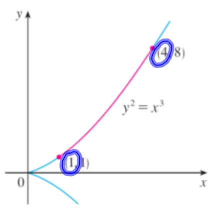
Calc: $\int_1^2 \sqrt{1+(y')^2}\,dy$

$\int_1^2 \sqrt{1+4}\,dx = \sqrt{5}\int_1^2 1\,dx$

$= \sqrt{5}\, x\Big|_1^2 = \sqrt{5}$

Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (See Figure 5.)

a) Set-up the integral
b) Calculate



$\int_a^b \sqrt{1+(y')^2}\,dx$

$a = 1$

$b = 4$

$y = x^{3/2}$

$y' = \frac{3}{2}x^{1/2}$

$(y')^2 = \frac{3}{2}x^{1/2}\,\frac{3}{2}x^{1/2} = \frac{9}{4}x^1$

Set-up integral: $\int_{x=1}^{x=4} \sqrt{1+\frac{9}{4}x}\,dx$

$u = 1+\frac{9}{4}x$

$du = \frac{9}{4}dx$

$dx = \frac{4}{9}du$

$u = 1+\frac{9}{4}x$

$x = 1 \Rightarrow u = 1+\frac{9}{4} = \frac{13}{4}$

$x = 4 \Rightarrow u = 1+\frac{9}{4}\cdot 4 = 10$

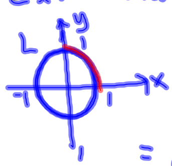
$\frac{4}{9}\int_{u=13/4}^{u=10} \sqrt{u}\,du$

$\frac{2}{3}\,\frac{4}{9}\,u^{3/2}\Big|_{13/4}^{10}$

$\frac{8}{27}\left(10^{3/2} - \frac{13}{4}^{3/2}\right) = \ldots\ldots$? Calculators.

$y = \sqrt{u} \Rightarrow y' = u'\frac{1}{2\sqrt{u}} = \frac{u'}{2\sqrt{u}}$

ex. Calculate arclength of circle $x^2+y^2 = 1$



$L = 2\pi r = 2\pi\cdot 1 = 2\pi$ ✓ (Geometry)

$4\int_0^1 \sqrt{1+(y')^2}\,dx$

$= 4\int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$ $\left(1 = \frac{1-x^2}{1-x^2}\right)$

$= 4\int_0^1 \sqrt{\frac{1}{1-x^2}}$

$= 4\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

$y^2 = 1-x^2$

$y = \sqrt{1-x^2}$

$y' = -2x\cdot\frac{1}{2\sqrt{1-x^2}}$

$(y')^2 = \frac{x^2}{1-x^2}$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$$4\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$4\int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = 4\int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\cos\theta} = 4\int_0^{\pi/2} 1\cdot d\theta$$

$x = \sin\theta$

$x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

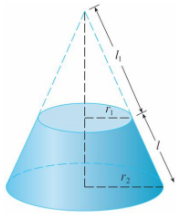
$$= 4\cdot\theta\Big|_0^{\pi/2}$$

$$= 4\cdot(\pi/2) = 2\pi$$

(1) Set up the Formula!

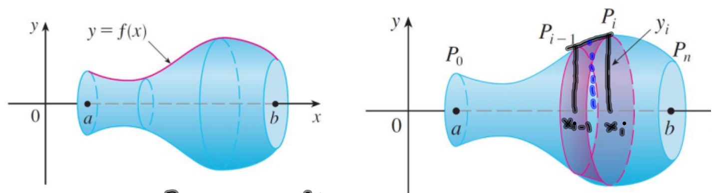
(2) Calculate arclength!

Next time: 8.2 ✓



2 $A = 2\pi r l$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.



$S = 2\pi \frac{f(y_{i-1})+f(y_i)}{2} \cdot |P_{i-1}P_i|$

8.2

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

$$\int 2\pi f \sqrt{1+(f')^2}\,dx$$

$$S = \int_a^b 2\pi \underbrace{f(x)}_{y} \sqrt{1 + [f'(x)]^2}\,dx \qquad \text{G}x$$

$$S = \int 2\pi x \sqrt{1+(f')^2}\,dx \qquad \text{G}y$$

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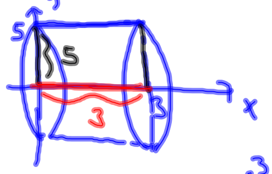
example: Find the surface area of the revolution obtained by rotating $y=5$ about x-axis, $0 < x < 3$



Geometry: $2\pi r h$

$2\pi\, 5\, 3 = 30\pi$

$y = 5$

$y' = 0$

Calculus 2: $\int_0^3 2\pi y \sqrt{1+(y')^2}\,dx$

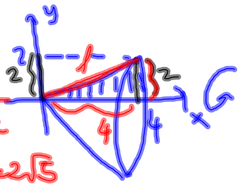
$S = \int_0^3 2\pi\, 5 \sqrt{1+0^2}\,dx = 10\pi \int_0^3 1\,dx = 10\pi\, x\Big|_0^3 = 30\pi$

ex. Find the surface area of the revolution obtained by rotating $y = \frac{x}{2}$, $0 \le x \le 4$

a) around x-axis

b) around y-axis.

a)



$l = \sqrt{4^2+2^2}$
$= \sqrt{20} = 2\sqrt{5}$

Geometry

$S = \pi r l = \pi \cdot 2 \cdot 2\sqrt{5}$

$= 4\sqrt{5}\pi$

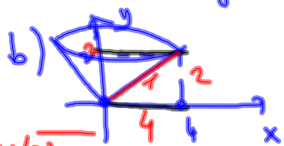
$y = \frac{x}{2} \quad y' = \frac{1}{2}$

$S = \int_0^4 2\pi y \sqrt{1+(y')^2}\, dx$

$S = 2\pi \int_0^4 \frac{x}{2} \sqrt{1+\frac{1}{4}}\, dx =$ $\qquad \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

$S = \pi \frac{\sqrt{5}}{2} \frac{x^2}{2}\Big|_0^4 = \pi \frac{\sqrt{5}}{2}(8) = 4\sqrt{5}\pi$

b)



$l = \sqrt{2^2+4^2}$
$= 2\sqrt{5}$

Geometry: $\pi r l$

$= \pi \, 4 \, 2\sqrt{5}$

$= 8\sqrt{5}\pi$

$y' = \frac{1}{2}$

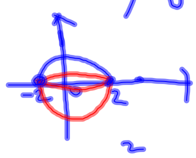
$S = \int 2\pi x \sqrt{1+(y')^2}\, dx$

$= 2\pi \int_0^4 x \sqrt{1+\frac{1}{4}}\, dx \qquad \sqrt{5}/2$

$= 2\pi \frac{\sqrt{5}}{2} \int_0^4 x\, dx = \pi\sqrt{5} \frac{x^2}{2}\Big|_0^4 = 8\sqrt{5}\pi$

example. Calculate surface area obtained rotating $y = \sqrt{4-x^2}$ around x-axis.

$y = \sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$



Geometry $S = 4\pi r^2$

$4\pi 2^2 = 16\pi$

Calculus 2: $\int_{-2}^{2} 2\pi y \sqrt{1+(y')^2}\, dx$

$y = \sqrt{(4-x^2)}$, u

$y = \sqrt{u}$

$y' = u' \cdot \frac{1}{2\sqrt{u}}$

$y' = -2x \cdot \frac{1}{2\sqrt{4-x^2}}$

$y' = \frac{-x}{\sqrt{4-x^2}}$

$(y')^2 = \frac{x^2}{4-x^2}$

$2\pi \int_{-2}^{2} \sqrt{4-x^2} \cdot \sqrt{1 + \frac{x^2}{4-x^2}}\, dx \qquad 1 = \frac{4-x^2}{4-x^2}$

$2\pi \int_{-2}^{2} \sqrt{4-x^2} \cdot \sqrt{\frac{4}{4-x^2}}\, dx = 2\pi \int_{-2}^{2} \sqrt{4-x^2} \cdot \frac{\sqrt{4}}{\sqrt{4-x^2}}\, dx$

$= 4\pi \int_{-2}^{2} 1 \cdot dx = 4\pi\, x \Big|_{-2}^{2} = 4\pi \cdot 4$

$= 16\pi$

Set-up the formula

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.

$S = \int_0^1 2\pi y \sqrt{1+(y')^2}\, dx$

$y = e^x$

$y' = e^x$

$(y')^2 = (e^x)^2 = e^x \cdot e^x = e^{2x}$

$(e^x)^2 \neq e^{x^2}$ ✗

$$S = \int_0^1 2\pi e^x \sqrt{1+e^{2x}}\, dx$$ ✓

$e^x = u \Rightarrow e^{2x} = u^2$

$u = e^x \quad du = e^x dx$

$S = 2\pi \int_1^e \sqrt{1^2+u^2}\, du \overset{7.3}{\Rightarrow} u = 1 \cdot \tan\theta \Rightarrow$ (cont.)

Ch 8

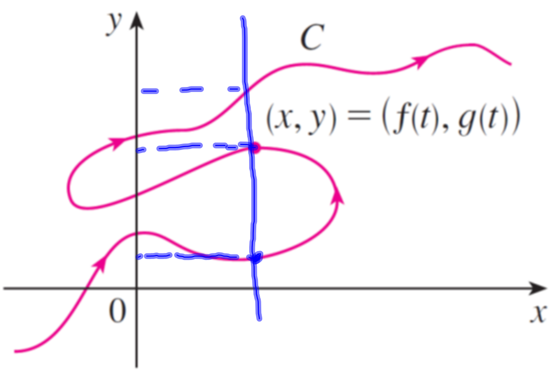
8.1 8.2

~~Chapter 9 (Diff. eq)~~

Chapter 10

Section 10.1 :

Curves Defined by Parametric Equations



$y = f(x)$ is not possible at all.

DEFINITION

Suppose $x = x(t)$ and $y = y(t)$ are two functions of a third variable t, called the **parameter**, that are defined on the same interval I. Then the equations

$$x = x(t) \qquad y = y(t)$$

where t is in I are called **parametric equations**, and the collection of points defined by

$$(x, y) = (x(t), y(t))$$

is called a **plane curve**.

(Parametric curve)

EXAMPLE 1 Graphing a Plane Curve

Graph the plane curve represented by the parametric equations

$$x(t) = 3t^2 \qquad y(t) = 2t \qquad -2 \le t \le 2$$

Indicate the orientation of the curve.

Solution Corresponding to each number t, $-2 \le t \le 2$, there are a number x and a number y that are the coordinates of a point (x, y) on the curve. We form a table listing various choices of the parameter t and the corresponding values for x and y, as shown in Table 1.

The motion begins when $t = -2$ at the point $(12, -4)$ and ends when $t = 2$ at the point $(12, 4)$. Figure 2 illustrates the plane curve whose parametric equations are $x(t) = 3t^2$ and $y(t) = 2t$. The arrows indicate the orientation of the plane curve for increasing values of the parameter t. ■

TABLE 1

t	x	y	(x, y)
−2	12	−4	(12, −4)
−1	3	−2	(3, −2)
0	0	0	(0, 0)
1	3	2	(3, 2)
2	12	4	(12, 4)

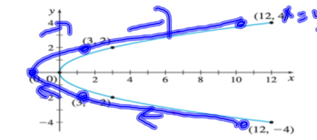


Figure 2 $x(t) = 3t^2$, $y(t) = 2t$, $-2 \le t \le 2$

$y = x^2$ $x = y^2$

$y = -x^2$ $x = -y^2$

$x = 3t^2$ $y = 2t \Rightarrow \frac{y}{2} = t$

$x = 3 \cdot \frac{y^2}{4} \rightarrow x = \frac{3}{4}y^2$

(Eliminate t if possible, and find a relation between x and y)

ex. eliminate t and find a relation between x and y

$x = t^4 - t^7 + t^2$

$y = \sin t + t^3 - t + e^t$

→ not possible

What curve is represented by the following parametric equations?

$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$

$x^2 = \cos^2 t \quad y^2 = \sin^2 t$

$x^2 + y^2 = 1$

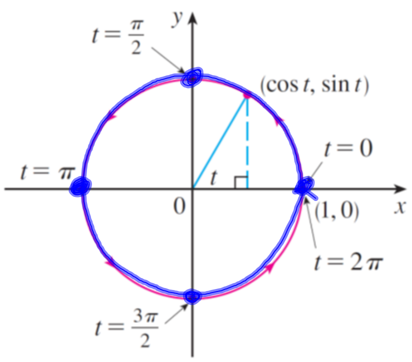
$x^2 = \cos^2 t$

$y^2 = \sin^2 t$

$+$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$t = \pi \quad x = \cos\pi = -1 \quad y = \sin\pi = 0$



A circle

$r = 1$, $O = (0,0)$ = Center

EXAMPLE 5 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

$y = x^2$ $\quad$ $y = \sin t \cdot \sin t = x \cdot x = x^2$

SOLUTION Observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \leqslant \sin t \leqslant 1$, we have $-1 \leqslant x \leqslant 1$, so the parametric equations represent only the part of the parabola for which $-1 \leqslant x \leqslant 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$. (See Figure 7.) ■ $(t \in \mathbb{R})$

$-1 \le \sin t \le 1 \Rightarrow -1 \le x \le 1$

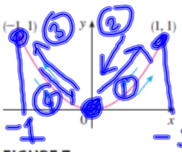


FIGURE 7

$t \in [0, 2\pi]$

	$x = \sin t$	$y = \sin^2 t$
$t = 0$	0	0
$t = \pi/2$	1	1
$t = \pi$	0	0
$t = 3\pi/2$	−1	1
$t = 2\pi$	0	0

Describe the motion of an object that moves along a curve so that at time t it has coordinates

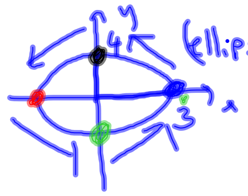
$x(t) = 3\cos t \quad y(t) = 4\sin t \quad 0 \le t \le 2\pi$

$\frac{x}{3} = \cos t \quad \frac{y}{4} = \sin t \quad t \in [0, 2\pi]$

$\frac{x^2}{9} = \cos^2 t$, $\frac{y^2}{16} = \sin^2 t$

$\frac{x^2}{9} + \frac{y^2}{16} = \cos^2 t + \sin^2 t = 1$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$ (ellipse)



$x = 3\cos t$, $y = 4\sin t \Rightarrow t = 0\ (3, 0)$

$t = \pi/2\ (0, 4)$

$t = \pi\ (-3, 0)$

$t = 3\pi/2\ (0, -4)$

(a) Find a rectangular equation of the plane curve whose parametric equations are

$x(t) = \cos(2t) \quad y(t) = \sin t \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

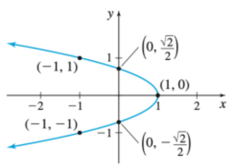
(b) Graph the rectangular equation.

$\sin^2 t = \frac{1 - \cos 2t}{2} \Rightarrow y^2 = \frac{1 - x}{2}$

$2y^2 = 1 - x$

$x = 1 - 2y^2$

$y = x^2$, $x = y^2$; $y = -x^2$, $x = -y^2$



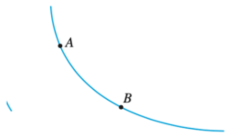
(a) $y^2 = \frac{1 - x}{2}$

THEOREM Parametric Equations of a Cycloid

Parametric equations of a cycloid are

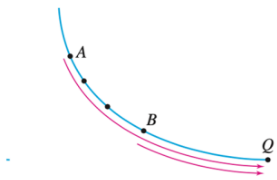
$$x(t) = a(t - \sin t) \qquad y(t) = a(1 - \cos t) \qquad -\infty < t < \infty$$

The **brachistochrone** is the curve of quickest descent. If an object is constrained to follow some path from a point A to a lower point B (not on the same vertical line) and is acted on only by gravity, the time needed to make the descent is minimized if the path is an inverted cycloid. See Figure 11(b). For example, in sliding packages from a loading dock onto a ship, a ramp in the shape of an inverted cycloid might be used so the packages get to the ship in the least amount of time. This discovery, which is attributed to many famous mathematicians (including Johann Bernoulli and Blaise Pascal), was a significant step in creating the branch of mathematics known as the *calculus of variations*.



(b) Curve of quickest descent

Suppose Q is the lowest point on an inverted cycloid. If several objects placed at various positions on an inverted cycloid simultaneously begin to slide down the cycloid, they will reach the point Q at the same time, as indicated in Figure 11(c). This is referred to as the **tautochrone property** of the cycloid. It was used by the Dutch mathematician,



(b) All particles reach Q at the same time

10.2 (Calculus with Parametric Curves)

DEFINITION Smooth Curve

Let C denote a plane curve represented by the parametric equations

$$x = x(t) \qquad y = y(t) \quad a \le t \le b$$

Suppose each function $x(t)$ and $y(t)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous and are never simultaneously 0 on (a, b), then C is called a **smooth curve**.

Example 10.2.12 Determine where a curve is not smooth

Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

Solution We begin by taking derivatives.

$$x' = 3t^2 - 12, \quad y' = 2t - 4.$$

$\frac{dx}{dt} = 3t^2 - 12$

$\frac{dy}{dt} = 2t - 4$

We set each equal to 0:

$$x' = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2$$
$$y' = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$$

We consider only the value of $t = 2$ since both x' and y' must be 0. Thus C is not smooth at $t = 2$, corresponding to the point $(1, 4)$. The curve is graphed in Figure 10.2.9, illustrating the cusp at $(1, 4)$.

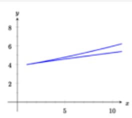


Figure 10.2.9: Graphing the curve in Example 10.2.12; note it is not smooth at $(1, 4)$.

If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change f x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some bject, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving bjects tend to change direction in a "smooth" fashion.

$\frac{dx}{dt} = 3t^2 - 12 = 0 \Rightarrow 3t^2 = 12 \Rightarrow t^2 = 4 \Rightarrow t = 2, -2$

$\frac{dy}{dt} = 2t - 4 = 0 \Rightarrow t = 2$

if $t = 2 \Rightarrow \frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero

This curve is smooth everywhere except $t = 2$.

at $t = 2 \Rightarrow \frac{dx}{dt} = 0$ $\frac{dy}{dt} = 0$

bject, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving bjects tend to change direction in a "smooth" fashion.

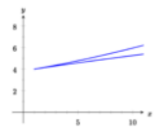


Figure 10.2.9: Graphing the curve in Example 10.2.12; note it is not smooth at (1, 4).

A smooth curve $x = x(t)$, $y = y(t)$, for which $\frac{dx}{dt}$ is never 0, can be represented by the rectangular equation $y = F(x)$, where F is differentiable. (You are asked to prove this in Problem 64.) Suppose $(x(t), y(t))$ is a point on the curve. Then

$$y = F(x)$$

$$y(t) = F(x(t))$$

Now, we use the Chain Rule to obtain

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Since $\frac{dx}{dt} \neq 0$, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Since $\frac{dy}{dx}$ is the slope of the tangent line to the graph of $y = F(x)$, we have proved the following result.

$x = x(t)$ $y = y(t)$ $\Rightarrow y = F(x)$

$y(x(t)) = F(x(t))$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ — Chain rule -

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

THEOREM Slope of the Tangent Line to a Smooth Curve

For a smooth curve C represented by the parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, the slope of the tangent line to C at the point (x, y) is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{1}$$

provided $\frac{dx}{dt} \neq 0$.

- At a number t where $\frac{dx}{dt} = 0$, but $\frac{dy}{dt} \neq 0$, a smooth curve C has a **vertical tangent line**.
- At a number t where $\frac{dy}{dt} = 0$, but $\frac{dx}{dt} \neq 0$, a smooth curve C has a **horizontal tangent line**.

(a) Find an equation of the tangent line to the plane curve with parametric equations $x(t) = 3t^2$, $y(t) = 2t$, when $t = 1$.

(b) Find all the points on the plane curve at which the tangent line is vertical.

---

a) Calculus 2: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{6t}$

$t=1 \Rightarrow$ slope $= \frac{2}{6 \cdot 1} = \frac{2}{6} = \frac{1}{3}$.

$y - y_0 = m(x - x_0)$, $m = 1/3$

$t=1 \Rightarrow x = 3 = x_0$, $y = 2 = y_0$

$y - 2 = \frac{1}{3}(x-3)$

$3y - 6 = x - 3$

$3y = x + 3$

$\boxed{y = \frac{1}{3}x + 1}$

Calculus 1: $x = 3t^2$, $y = 2t$, $t = \frac{y}{2}$ $\Rightarrow$ $x = \frac{3}{4} \cdot y^2$

$4x = 3y^2$

$\boxed{y^2 = \frac{4x}{3}}$

$x = 3$, $y = 2$

$y = 2\sqrt{\frac{x}{3}} = \frac{2}{\sqrt{3}}\sqrt{x}$

$y' = \frac{2}{\sqrt{3}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{3} \cdot \sqrt{x}}$

$x = 3$: $y' = m = \text{slope} = \frac{1}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}$

$y - 2 = \frac{1}{3}(x-3) \Rightarrow$ $\boxed{y = \frac{1}{3}x + 1}$

b) Calculus 2: $\frac{dy}{dx} = \frac{2}{6t}$ = SLOPE

$t = 0 \Rightarrow m$ is undefined

$x = 3t^2 \quad y = 2t \Rightarrow (0,0)$

Calculus 1: $\frac{dy}{dx} = \frac{1}{\sqrt{3}\cdot\sqrt{x}}$

$x = 3t^2 = 0 \Rightarrow t = 0$

$y = 2t$

$y = 0$

$x = 0 \Rightarrow$ undefined

$(0,0)$

Sunday

✓ Tuesday → B1-004 9–12

Monday: 10.2, I answer your questions

Wednesday: 12-1 exam.

Quiz 2

$$\frac{\quad}{dx} = \frac{\frac{\quad}{dt}}{\frac{dx}{dt}} \tag{1}$$

ical

izontal

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point (3, 0) and find their equations.

$(3,0) \rightarrow t = ?$
x, y

$x = t^2 = 3 \Rightarrow t^2 = 3 \Rightarrow t = -\sqrt{3}, \sqrt{3}$

$y = t^3 - 3t = 0 \Rightarrow t.(t^2-3) = 0$

~~$t \neq 0$~~ $t = -\sqrt{3}, \sqrt{3}$.

$t = \sqrt{3}, t = \sqrt{3}$

~~The graph crosses itself at (3,0).~~

$y - y_0 = m(x - x_0)$, $(x_0, y_0) = (3,0)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} = \text{SLOPE}$$

$t = \sqrt{3}$

$$\text{SLOPE} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

($3 \rightarrow \sqrt{3}.\sqrt{3}$)

$y - 0 = \sqrt{3}(x-3)$

$$\boxed{y = \sqrt{3}(x-3)}$$

$t = -\sqrt{3}$

$$\text{SLOPE} = \frac{6}{2\sqrt{3}} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$$

$y - 0 = -\sqrt{3}(x-3)$

$$\boxed{y = -\sqrt{3}(x-3)}$$

(b) Find the points on C where the tangent is horizontal or vertical.

$x = t^2$
$y = t^3 - 3t$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t}$$

Horizontal: $3t^2 - 3 = 0 \Rightarrow 3t^2 = 3 \Rightarrow t^2 = 1 \Rightarrow t = -1, 1$ → $(1, 2)$, $(1, -2)$

Vertical: $2t = 0$, $t = 0$ → $x = 0, y = 0$ $(0,0)$

(c) Determine where the curve is concave upward or downward.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{6t.2t - 2(3t^2-3)}{4t^2.2t}$$

$$= \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{6t^2+6}{8t^3}$$

($6t^2 + 6 > 0$)

$t > 0$: $y'' > 0 \Rightarrow$ up. ∪

$t < 0$: $y'' < 0 \Rightarrow$ down. ∩

(d) Sketch the curve.

$t=-\sqrt{3}$, $t=\sqrt{3}$ → $y=\sqrt{3}(x-3)$; $(3,0)$: Graph crosses itself at $(3,0)$; $y=-\sqrt{3}(x-3)$

$(0,0)$ → vertical, $t=0$

$t=-1 \Rightarrow (1,2)$, $t=1 \Rightarrow (1,-2)$ → horizontal

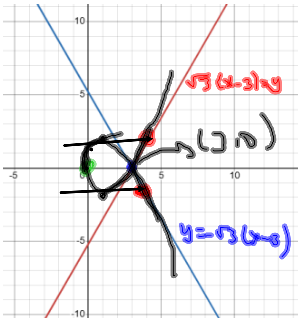
$t>0 \Rightarrow$ up

$t<0 \Rightarrow$ down

Extra points

$t=-2$: $(4,-2)$

$t=2$: $(4,2)$

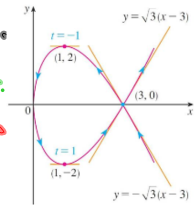


• $(3,0)$. Crosses it

• Horizon

• $(0,0)$:

• extra



Area problem: $\begin{cases} x=f(t) \\ y=g(t) \end{cases}$ is given → $dx = f'(t)\,dt$

$$A \overset{(6.1)}{=} \int_a^b y\,dx = \int_\alpha^\beta \underbrace{g(t)}_{y}\,\underbrace{f'(t)\,dt}_{dx} \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t\right.$$

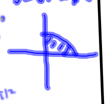
---

ex. Find the area of the circle $x^2+y^2=r^2$.

7.3

$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$



Solve the integral: Area $=4\int_0^r \sqrt{r^2-x^2}\,dx$, $x=r\sin\theta$, $dx=r\cos\theta\,d\theta$

$=4r^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 4r^2\cdot\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$=\pi r^2$

Trigonometric Subst.

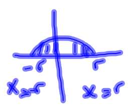
Trig Sub + $\int \cos^2 t$

7.3 + 7.2

10.2

$x=r\cos t = f(t)$

$y=r\sin t = g(t)$



Area $=2\int r\sin t \cdot -r\sin t$

$\int_{-r}^{r}$ $= -2r^2\int \sin^2 t\,dt$

$x = r\cos t$

$-r = r\cos t \Rightarrow \cos t=-1 \Rightarrow t=\pi$

$r = r\cos t \Rightarrow \cos t = 1 \Rightarrow t=0$

$A = -2r^2\int_\pi^0 \left(\frac{1-\cos 2t}{2}\right)dt$

$A = -r^2\left(t-\frac{1}{2}\sin 2t\right)_\pi^0$

$A = -r^2\left((0-0)-(\pi-0)\right)$

$= -r^2\cdot(0-\pi) = \pi r^2$

$\int \sin^2 t$ (7.2)

Wednesday: 9-11 S1.36 12-1 exam

Monday:

Wednesday:

} 10.2 + 10.3 + 10.4

Find the area under one arch of the cycloid

$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$

$dx = r(1-\cos\theta)\,d\theta$

(See Figure 3.)

$x = r(\theta - \sin\theta)$

$0 = r.(\theta - \sin\theta)$, $= 0$

$\sin\theta = \theta$

$\theta = 0.$

$2\pi r = r(\theta - \sin\theta)$

$2\pi = \theta - \sin\theta . \quad \theta = 2\pi$

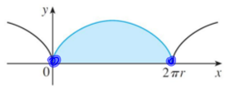


Figure 3

$$\int_{x=0}^{x=2\pi r} y\,dx = \int_{\theta=0}^{\theta=2\pi} r(1-\cos\theta) . r(1-\cos\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2\,d\theta \quad \checkmark$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \cos^2\theta\right] d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= r^2 \left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}$$

$$= r^2 \left[(3\pi) - (0)\right] = \boxed{3\pi r^2.}$$

Arclength

$(8.1) \Rightarrow \quad L = \int_a^b \sqrt{1 + (y')^2}\,dx .$

$\boxed{\begin{array}{l} x = x(t) \\ y = y(t) \end{array}} \Rightarrow y' = \frac{dy/dt}{dx/dt}$

$x = x(t)$

$dx = x'(t)dt = \frac{dx}{dt}dt$

$$L = \int \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}} \cdot \frac{dx}{dt}.dt$$

$$L = \int \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}} \cdot \frac{dx}{dt}\,dt$$

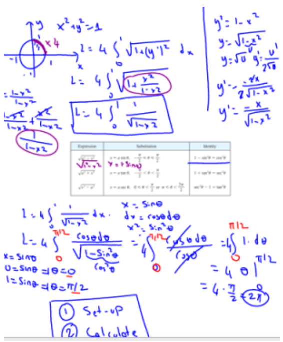
$$\boxed{L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

---

ex - Calculate arclength of the circle:

8.1

$x^2 + y^2 = r^2$



10.2

$x = r\cos t$

$y = r\sin t$ $\quad 0 \le t \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\,dt$$

$x' = -r\sin t \qquad y' = r\cos t$

$(x')^2 = r^2\sin^2 t \qquad (y')^2 = r^2\cos^2 t$

$$L = \int_0^{2\pi} \sqrt{r^2(\cos^2 t + \sin^2 t)}\,dt$$

$$= r\int_0^{2\pi} 1\,dt = r\,t\Big|_0^{2\pi} = 2\pi r$$

Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$. , $0 < \theta < 2\pi$.

$x' = r(1-\cos\theta) \Rightarrow (x')^2 = r^2(1-\cos\theta)^2 = r^2(1 - 2\cos\theta + \cos^2\theta)$

$(a-b)^2 = a^2 - 2ab + b^2$

$y' = r\cdot\sin\theta \Rightarrow (y')^2 = r^2\sin^2\theta$.

$$L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\,dt = \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\,d\theta$$

$$= r\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,d\theta \qquad 2(1-\cos\theta)$$

$$= r\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$2\sin^2\theta = 1 - \cos 2\theta$

$\theta \to \frac{\theta}{2}$

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

$$= r\sqrt{2}\int_0^{2\pi} \left(\sqrt{2\sin^2\frac{\theta}{2}}\right)d\theta \qquad \sqrt{2}\cdot\sqrt{\sin^2\frac{\theta}{2}}$$

$$= 2r\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta$$

$$= 2r\cdot -2\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= -4r\left[-1-1\right]$$

$$= -4r\cdot -2 = 8r.$$

6.1 ✓

8.1 ✓

8.2 ✓

Surface Area

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In **Curve Length and Surface Area**, we derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the x-axis:

$$S = 2\pi\int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx. \quad (8.2)$$

$f(x) = y = y(t)$; $\sqrt{1 + (f'(x))^2} \rightarrow \sqrt{(x')^2 + (y')^2}$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t),\ y = y(t),\ a \le t \le b$ around the x-axis as shown in the following figure.

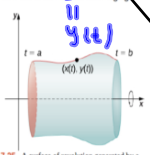


Figure 7.25 A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S = 2\pi\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \qquad (7.6)$$

provided that $y(t)$ is not negative on $[a, b]$.

THEOREM Surface Area of a Solid of Revolution

The surface area S of a solid of revolution generated by revolving the smooth curve C represented by $x = x(t)$, $y = y(t)$, $a \le t \le b$, where $y(t) \ge 0$, about the x-axis is

$$S = 2\pi\int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad (2)$$

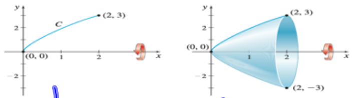
THEOREM Surface Area of a Solid of Revolution

The surface area S of a solid of revolution generated by revolving the smooth curve C represented by $x = x(t)$, $y = y(t)$, $a \le t \le b$, where $x = x(t) \ge 0$, about the y-axis is

$$S = 2\pi\int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Find the surface area of the solid generated by revolving the smooth curve C represented by the parametric equations $x(t) = 2t^3$, $y(t) = 3t^2$, $0 \le t \le 1$, about the x-axis.

Solution We begin by graphing the smooth curve C and revolving it about the x-axis. See Figure 22.



$x' = 6t^2$, $(x')^2 = 36t^4$

$y' = 6t$, $(y')^2 = 36t^2$

$S = 2\pi \int_0^1 y\sqrt{(x')^2+(y')^2}\,dt$

$S = 2\pi \int_0^1 3t^2\left(\sqrt{36t^4+36t^2}\right)dt$ → $\sqrt{36t^2}\cdot\sqrt{t^2+1}$

$= 2\pi \int_0^1 3t^2 \cdot 6t \cdot \sqrt{1+t^2}\,dt$

$= 36\pi \int_0^1 t^3\sqrt{1+t^2}\,dt$

$u = 1+t^2$, $du = 2t\,dt$, $\frac{1}{2}du = t\,dt$

$= 36\pi \int_1^2 (u-1)\sqrt{u}\,\frac{1}{2}du$

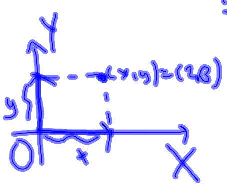
$= 18\pi \int_1^2 (u^{3/2} - u^{1/2})\,du$

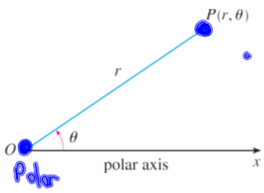
$= 18\pi\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^2 =$

when t

$\frac{24\pi}{5}(\sqrt{2}+1)$

Rectangular 10.3 Polar





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(1)

If $P = O$, then $r = \theta$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

(2)

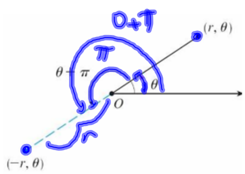


Figure 2

(3) $(-r, \theta) = (r, \pi + \theta)$

(4)

Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

(r, θ)

Solution:

The points are plotted in Figure 3.

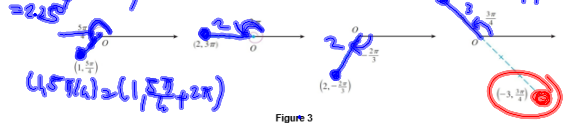


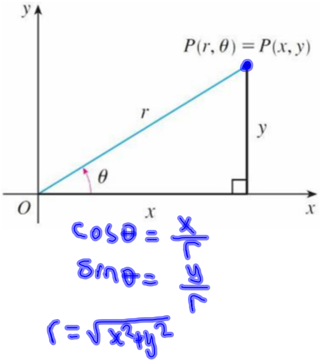
Figure 3

(Next TIME)

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.



Polar to rectangular

(r,θ) is given: $(x,y) = ?$

$x = r\cos\theta$

$y = r\sin\theta$

(x,y) is given: $(r,\theta) = ?$

$r = \sqrt{x^2+y^2}$

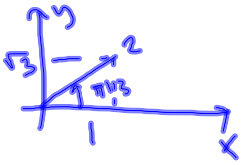
$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates. (r, θ)

$x = r\cos\theta = 2\cos\pi/3 = 2\cdot\frac{1}{2} = 1$

$y = r\sin\theta = 2\cdot\sin\pi/3 = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

$\Rightarrow (1, \sqrt{3})$



Represent the point with Cartesian coordinates (1, –1) in terms of polar coordinates. (x,y)

$r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$.

$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 \Rightarrow \tan\theta = -1 \Rightarrow \theta =$

$\theta = \arctan(-1)$

Common mistake: $3\pi/4$

$7\pi/4$

Table of Trigonometric Functions – Exact Values for Special Angles

Angle θ in degrees	Angle θ in radians	sin(θ)	cos(θ)	tan(θ)	cot(θ)	sec(θ)	csc(θ)
0°	0	0	1	0	undef.	1	undef.
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	undef.	0	undef.	1
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	−2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	−1	−1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180°	π	0	−1	0	undef.	−1	undef.
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	−2
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	−2	$-\frac{2\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	−1	0	undef.	0	undef.	−1
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	−1	−1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	−2
360°	2π	0	1	0	undef.	1	undef.

Example 5: Convert the rectangular equation $x^2 + y^2 = 100$ into a polar equation that expresses r in terms of θ.

Solution: $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$ $r^2\cos^2\theta + r^2\sin^2\theta = 100$

$r^2(\cos^2\theta + \sin^2\theta) = 100$

$r^2 = 100 \Rightarrow$ $r = 10$, $r = -10$

Substitute x and y with their polar equivalents

$x = r\cos\theta$ and $y = r\sin\theta$

$x^2 + y^2 = 100$

$(r\cos\theta)^2 + (r\sin\theta)^2 = 100$

Solve for r

$r^2\cos^2\theta + r^2\sin^2\theta = 100$

$r^2(\cos^2\theta + \sin^2\theta) = 100$ factor common term r^2

$r^2(1) = 100$ apply Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$

$r^2 = 100$

$r = 10$ ✓

$r = -10$ ✓

Example 6: Convert the polar equation 4r cos θ + r sin θ = 8 into a rectangular equation that expresses y in terms of x.

x y

Solution:

Substitute r cos θ and r sin θ with their rectangular equivalents

4r cos θ + r sin θ = 8
4(x) + (y) = 8

$4x + y = 8$

Solve for y

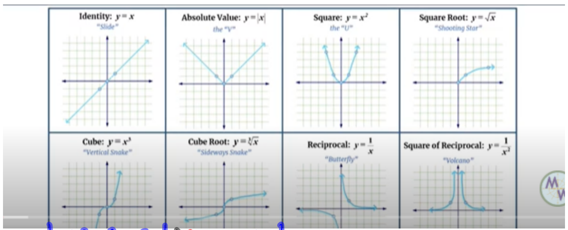
4x + y = 8
y = 8 – 4x

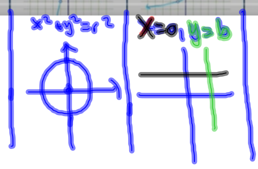
$y = -4x + 8$

Line, slope = −4.

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Basic Graphs in rectangular





WHAT ARE BASIC GRAPHS IN POLAR

EXAMPLE 5 Identifying and Graphing a Polar Equation

Identify and graph each equation:

(a) $r = 3$ **(b)** $\theta = \frac{\pi}{4}$

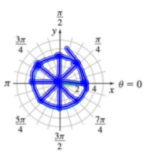


Figure 39 $r = 3$ or $x^2 + y^2 = 9$.

Solution (a) If r is fixed at 3 and θ is allowed to vary, the graph is a circle with its center at the pole and radius 3, as shown in Figure 39. To confirm this, we convert the polar equation $r = 3$ to a rectangular equation.

$$r = 3$$
$$r^2 = 9 \quad \text{Square both sides.}$$
$$x^2 + y^2 = 9 \quad r^2 = x^2 + y^2$$

$r = 3$
$\sqrt{x^2+y^2} = 3$
$x^2+y^2 = 9$

(b) If θ is fixed at $\frac{\pi}{4}$ and r is allowed to vary, the result is a line containing the pole, making an angle of $\frac{\pi}{4}$ with the polar axis. That is, the graph of $\theta = \frac{\pi}{4}$ is a line containing the pole with slope $\tan\theta = \tan\frac{\pi}{4} = 1$, as shown in Figure 40. To confirm this, we convert the polar equation to a rectangular equation.

$$\theta = \frac{\pi}{4}$$
$$\tan\theta = \tan\frac{\pi}{4}$$
$$\frac{y}{x} = 1$$
$$y = x$$

$\theta = \pi/4$
$\tan\theta = \tan\pi/4$
$\frac{y}{x} = 1$
$y = x$ ✓

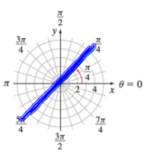


Figure 40 $\theta = \frac{\pi}{4}$ or $y = x$.

In general, the equation $r = a$ represents a circle with center O and radius $|a|$.

(a) $r \sin\theta = 2$

Solution (a) Here, bot

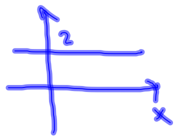
$r\cos\theta = 2$

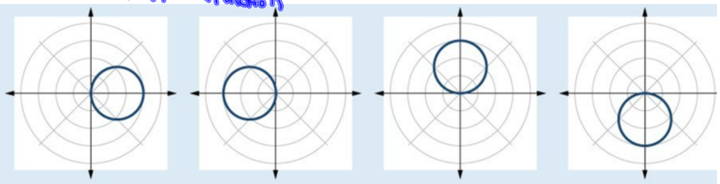
Vertical line:

mcq. $r\sin\theta = 2$ is
(a) vertical line
b) horizontal line
c) a circle d) cubic function

$y = 2$

$x = 2$ Horizontal line





$r = a\cos\theta,\ a > 0$ (a) | $r = a\cos\theta,\ a < 0$ (b) | $r = a\sin\theta,\ a > 0$ (c) | $r = a\sin\theta,\ a < 0$ (d)

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, where a is t diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the dia eter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 5 shows the graphs of these four circles.

(b) $r = 4\sin\theta$

oth r and θ are allowed to

mcq: (or convert from polar to rectangular) $r = 4\sin\theta$ is
a) a line
b) a parabola
c) a circle $r = 16$ center (0,0)
(d) a circle $r = 2$ center (0,2)
e) none of given

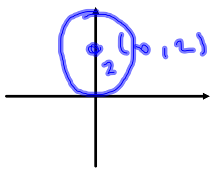
$r^2 = x^2 + y^2$
$r\cos\theta = x$
$r\sin\theta = y$

$(y-2)^2 = y^2 - 4y + 4$

$r \cdot r = 4\sin\theta \cdot r$
$r^2 = 4r\sin\theta$
$x^2 + y^2 = 4y$
$x^2 + y^2 - 4y = 0$
$x^2 + (y-2)^2 - 4 = 0$
$x^2 + (y-2)^2 = 2^2$

$(x-a)^2 + (y-b)^2 = r^2$
center: (a, b)
radius $= r$

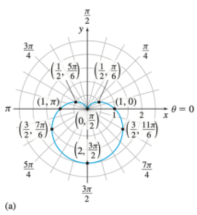
center: $(0, 2)$
radius: 2



Solution (a) The polar equation $r = 1 - \sin\theta$ contains $\sin\theta$, which has the period 2π. We construct Table 2 using common values of θ that range from 0 to 2π, plot the points (r, θ), and trace out the graph, beginning at the point $(1, 0)$ and ending at the point $(1, 2\pi)$, as shown in Figure 43(a). Figure 43(b) shows the graph using technology.

TABLE 2

θ	$r = 1 - \sin\theta$	(r, θ)
0	$1 - 0 = 1$	$(1, 0)$
$\frac{\pi}{6}$	$1 - \frac{1}{2} = \frac{1}{2}$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$
$\frac{\pi}{2}$	$1 - 1 = 0$	$\left(0, \frac{\pi}{2}\right)$
$\frac{5\pi}{6}$	$1 - \frac{1}{2} = \frac{1}{2}$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$
π	$1 - 0 = 1$	$(1, \pi)$
$\frac{7\pi}{6}$	$1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$	$\left(\frac{3}{2}, \frac{7\pi}{6}\right)$
$\frac{3\pi}{2}$	$1 - (-1) = 2$	$\left(2, \frac{3\pi}{2}\right)$
$\frac{11\pi}{6}$	$1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$	$\left(\frac{3}{2}, \frac{11\pi}{6}\right)$
2π	$1 - 0 = 1$	$(1, 2\pi)$



(a)

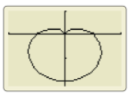


Figure 43 The cardioid $r = 1 - \sin\theta$.

NOTE Graphs of polar equations of the form

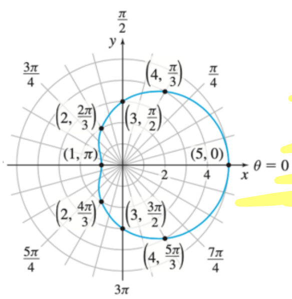
$r = a(1 + \cos\theta)$ $\quad r = a(1 + \sin\theta)$
$r = a(1 - \cos\theta)$ $\quad r = a(1 - \sin\theta)$

where $a > 0$ are called **cardioids**. A cardioid contains the pole and is heart-shaped (giving the curve its name).

(a) Graph the polar equation $r = 3 + 2\cos\theta, 0 \le \theta \le 2\pi$.

TABLE 3

θ	$r = 3 + 2\cos\theta$	(r, θ)
0	$3 + 2(1) = 5$	$(5, 0)$
$\frac{\pi}{3}$	$3 + 2\left(\frac{1}{2}\right) = 4$	$\left(4, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$3 + 2(0) = 3$	$\left(3, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$3 + 2\left(-\frac{1}{2}\right) = 2$	$\left(2, \frac{2\pi}{3}\right)$
π	$3 + 2(-1) = 1$	$(1, \pi)$
$\frac{4\pi}{3}$	$3 + 2\left(-\frac{1}{2}\right) = 2$	$\left(2, \frac{4\pi}{3}\right)$
$\frac{3\pi}{2}$	$3 + 2(0) = 3$	$\left(3, \frac{3\pi}{2}\right)$
$\frac{5\pi}{3}$	$3 + 2\left(\frac{1}{2}\right) = 4$	$\left(4, \frac{5\pi}{3}\right)$
2π	$3 + 2(1) = 5$	$(5, 2\pi)$

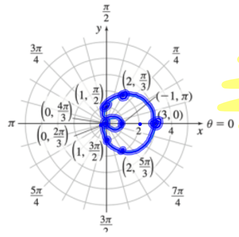


NOTE Graphs of polar equations of the form

$r = a + b\cos\theta$ $\quad r = a + b\sin\theta$
$r = a - b\cos\theta$ $\quad r = a - b\sin\theta$

where $a > b > 0$, are called **limaçons without an inner loop**. A limaçon without an inner loop does not pass through the pole.

θ	$r = 1 + 2\cos\theta$	(r, θ)
0	$1 + 2(1) = 3$	$(3, 0)$
$\frac{\pi}{3}$	$1 + 2\left(\frac{1}{2}\right) = 2$	$\left(2, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$1 + 2(0) = 1$	$\left(1, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$1 + 2\left(-\frac{1}{2}\right) = 0$	$\left(0, \frac{2\pi}{3}\right)$
π	$1 + 2(-1) = -1$	$(-1, \pi)$
$\frac{4\pi}{3}$	$1 + 2\left(-\frac{1}{2}\right) = 0$	$\left(0, \frac{4\pi}{3}\right)$
$\frac{3\pi}{2}$	$1 + 2(0) = 1$	$\left(1, \frac{3\pi}{2}\right)$
$\frac{5\pi}{3}$	$1 + 2\left(\frac{1}{2}\right) = 2$	$\left(2, \frac{5\pi}{3}\right)$
2π	$1 + 2(1) = 3$	$(3, 2\pi)$



NOTE Graphs of polar equations of the form

$r = a + b\cos\theta \quad r = a + b\sin\theta$

$r = a - b\cos\theta \quad r = a - b\sin\theta$

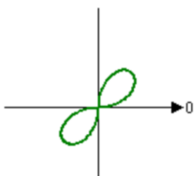
where $0 < a < b$, are called **limaçons with an inner loop**. A limaçon with an inner loop passes through the pole twice.

Lemniscates

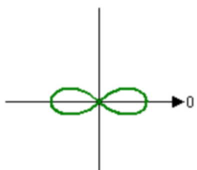
The last type of polar equation that we will cover here is the lemniscates, which has the shape of a figure-8 or a propeller. Lemniscates have the general polar equation of:

$$r^2 = a^2 \sin 2\theta \quad \text{or} \quad r^2 = a^2 \cos 2\theta, \text{ where } a \neq 0$$

A lemniscate containing the sine function will be symmetric to the pole while the lemniscate containing the cosine function will be symmetric to the polar axis, to $\theta = \frac{\pi}{2}$, and the pole.



$r^2 = a^2 \sin 2\theta$

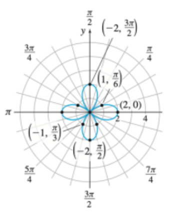


$r^2 = a^2 \cos 2\theta$

You have to know the equations of basic graphs in POLAR.

TABLE 5

θ	$r = 2\cos(2\theta)$	(r, θ)	θ	$r = 2\cos(2\theta)$	(r, θ)
0	$2(1) = 2$	$(2, 0)$			
$\frac{\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$	$\left(1, \frac{\pi}{6}\right)$	$\frac{7\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$	$\left(1, \frac{7\pi}{6}\right)$
$\frac{\pi}{4}$	$2(0) = 0$	$\left(0, \frac{\pi}{4}\right)$	$\frac{5\pi}{4}$	$2(0) = 0$	$\left(0, \frac{5\pi}{4}\right)$
$\frac{\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$	$\left(-1, \frac{\pi}{3}\right)$	$\frac{4\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$	$\left(-1, \frac{4\pi}{3}\right)$
$\frac{\pi}{2}$	$2(-1) = -2$	$\left(-2, \frac{\pi}{2}\right)$	$\frac{3\pi}{2}$	$2(-1) = -2$	$\left(-2, \frac{3\pi}{2}\right)$
$\frac{2\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$	$\left(-1, \frac{2\pi}{3}\right)$	$\frac{5\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$	$\left(-1, \frac{5\pi}{3}\right)$
$\frac{3\pi}{4}$	$2(0) = 0$	$\left(0, \frac{3\pi}{4}\right)$	$\frac{7\pi}{4}$	$2(0) = 0$	$\left(0, \frac{7\pi}{4}\right)$
$\frac{5\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$	$\left(1, \frac{5\pi}{6}\right)$	$\frac{11\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$	$\left(1, \frac{11\pi}{6}\right)$
π	$2(1) = 2$	$(2, \pi)$	2π	$2(1) = 2$	$(2, 2\pi)$



(b) $r = 2\cos(2\theta)$, $0 \le \theta \le 2\pi$

NOTE Graphs of polar equations of the form $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$, $a > 0$, n an integer, are called **roses**. If n is an even integer, the rose has $2n$ petals and passes through the pole $4n$ times. If n is an odd integer, the rose has n petals and passes through the pole $2n$ times.

odd number

mcq: $r = 3\cos(5\theta)$ is

(a) a rose with 5 petals
b) a rose with 10 petals
c) a line
d) a circle

Monday: 10.3 + 10.4

Wednesday: 9-11 in SG.38 (LOSGS)

"I will post a review in Moodle over the weekend"

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

In Final Exam

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

- SLOPE.
- Vertical / horizontal tangent line.

Then, using the method for finding slopes of parametric curves and the Product Rule, we have

$$\boxed{3} \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

a) Solution:

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\cdot\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\cdot\sin\theta} =$$

$$= \frac{\cos\theta(2\sin\theta+1)}{1-2\sin^2\theta-\sin\theta}$$

$\cos^2\theta - \sin^2\theta - \sin\theta$

$1 - 2\sin^2\theta$

$\Rightarrow \sin\theta = x$ $\qquad 1 - 2x^2 - x$ $\quad (1+x)(1-2x)$ ✓

$$= \frac{\cos\theta(2\sin\theta+1)}{(1+\sin\theta)(1-2\sin\theta)}$$

$\cos\pi/3 = 1/2$

$\sin\pi/3 = \sqrt{3}/2$

SLOPE $(\theta = \pi/3)$ $= \dfrac{\cos\pi/3(2\sin\frac{\pi}{3}+1)}{(1+\sin\pi/3)(1-2\sin\pi/3)}$

$$= \frac{\frac{1}{2}\left(2\frac{\sqrt{3}}{2}+1\right)}{\left(1+\frac{\sqrt{3}}{2}\right)\left(1-2\cdot\frac{\sqrt{3}}{2}\right)} = \frac{\frac{1}{2}(\sqrt{3}+1)}{\frac{2+\sqrt{3}}{2}(1-\sqrt{3})}$$

$[2 - 2\sqrt{3} + \sqrt{3} - 3$

$= -1 - \sqrt{3}$

$= -(1+\sqrt{3})]$

$$= \frac{\sqrt{3}+1}{-(1+\sqrt{3})} = -1.$$

SLOPE $= -1$.

---

b) $$\frac{dy}{dx} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

Table of Trigonometric Functions – Exact Values for Special Angles

Angle θ in degrees	Angle θ in radians	sin(θ)	cos(θ)	tan(θ)	cot(θ)	sec(θ)	csc(θ)
0°	0	0	1	0	undef.	1	undef.
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	undef.	0	undef.	1
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	−2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	−1	−1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180°	π	0	−1	0	undef.	−1	undef.
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	−2
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	−2	$-\frac{2\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	−1	0	undef.	0	undef.	−1
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	−1	−1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	−2
360°	2π	0	1	0	undef.	1	undef.

Horizontal: $\cos\theta = 0 \Rightarrow \pi/2, 3\pi/2$

$1 + 2\sin\theta = 0$ –

$\sin\theta = -1/2 \Rightarrow \frac{7\pi}{6}, \frac{11\pi}{6}$.

Vertical: $1 + \sin\theta = 0 \Rightarrow \sin\theta = -1 \Rightarrow 3\pi/2$

$1 - 2\sin\theta = 0 \Rightarrow \sin\theta = 1/2 \Rightarrow \frac{\pi}{6}, \frac{5\pi}{6}$.

$$\lim_{(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta}$$

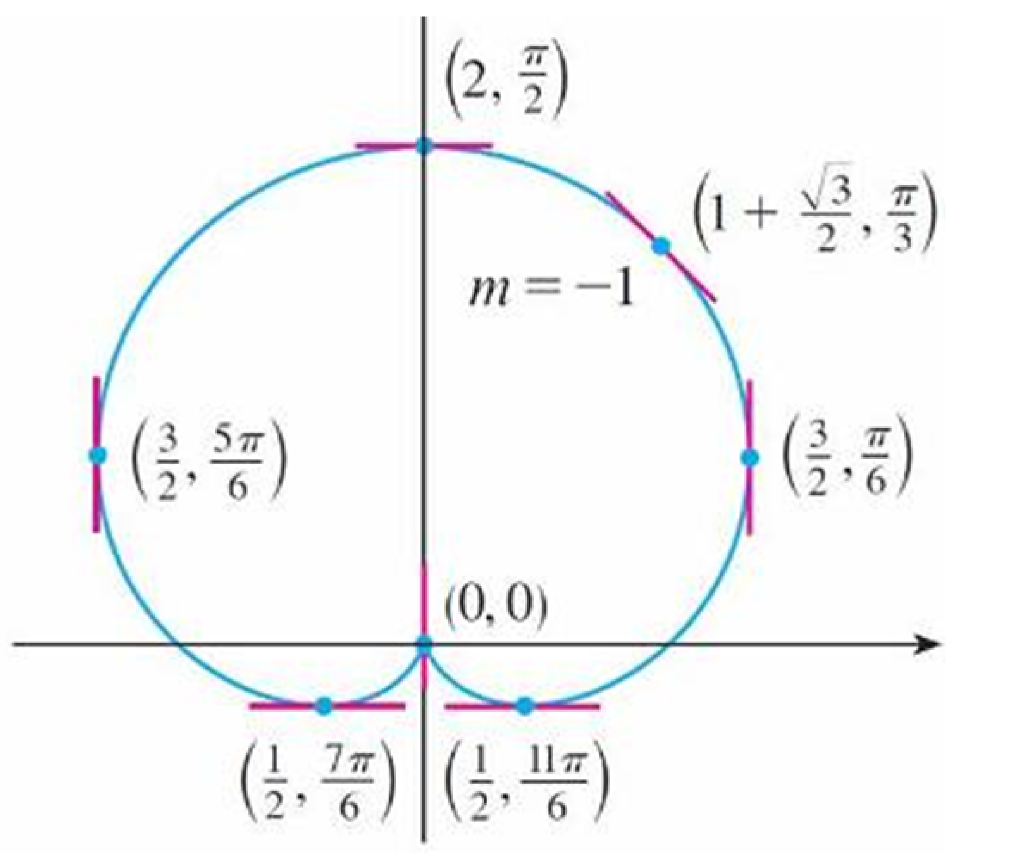
$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta}$$

$$= \infty$$

$\theta = 3\pi/2$: Vertical

By symmetry,

$$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$



10.4

Area :

$A = \frac{1}{2} r^2 \theta$

Geometry

is the radius and angle.

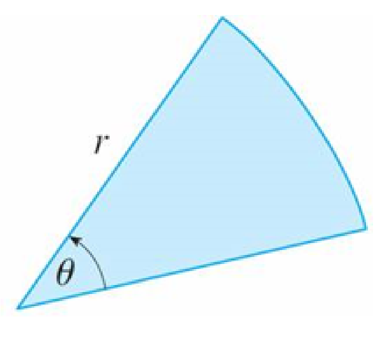
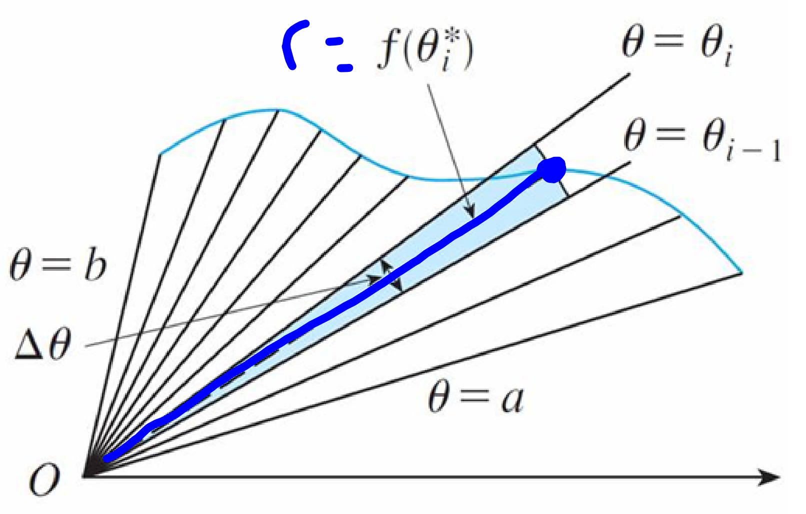


Figure 1



$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$ ($f(\theta_i^*)$ = r)

nation to the total area A of $\mathcal{R}$ is

$$A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta$$

3
$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta$$

Calculate area of the circle

$x^2 + y^2 = 1$ | $x = r\cos\theta$ $y = r\sin\theta$ | $r = a$

Solve the integral

Area $= 4\int_0^1 \sqrt{1-x^2}\,dx$ ... $= \pi r^2$

$$\frac{1}{2}\int_0^{2\pi} a^2\, d\theta$$

$$\frac{1}{2} a^2 \int_0^{2\pi} 1\, d\theta$$

$$\frac{1}{2} a^2 \,\theta \Big|_0^{2\pi}$$

$$\frac{1}{2} a^2 \cdot 2\pi = \pi a^2.$$

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$. , $-\pi/4 < \theta < \pi/4$.

Area $= \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$ (7.2)

Side work: $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
$\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{\theta=-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}+\frac{1}{4}\cdot 0\right) - \left(-\frac{\pi}{4}+\frac{1}{4}\cdot 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + \left(+\frac{\pi}{4}\right)\right] = \frac{1}{4}\cdot\frac{2\pi}{4} = \frac{2\pi}{16} = \frac{\pi}{8}.$$

Set up the integral to

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

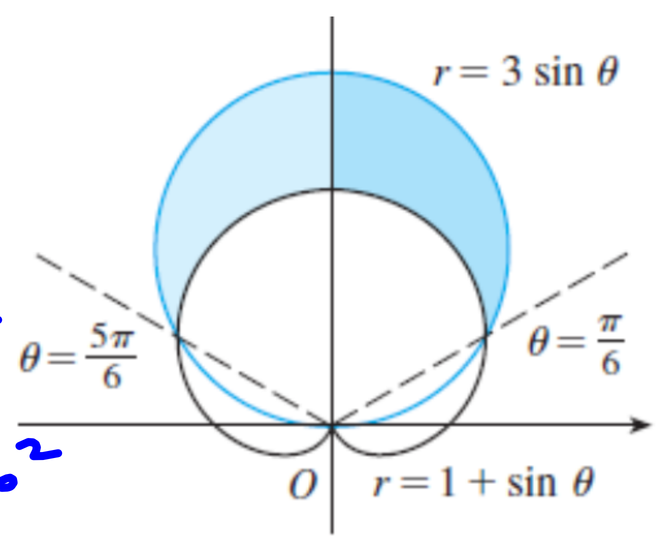


FIGURE 5

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2\, d\theta$$

$(a+b)^2 = a^2 + 2ab + b^2$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[8\sin^2\theta - 1 - 2\sin\theta\right] d\theta$$

Calculate

$$(7.2) - \theta + 2\cos\theta \Big|_{\pi/6}^{5\pi/6}$$

Arclength Problem

11.1

10.4

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \le \theta \le b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$(r')^2(\cos^2\theta + \sin^2\theta)$, $= 1$; $r^2(\cos^2\theta + \sin^2\theta)$, $= 1$

5

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Calculate arclength of the circle $r = 3$ ($r' = 0$)

$L = \int_0^{2\pi} \sqrt{9 + 0^2}\, d\theta = 3\int_0^{2\pi} 1\, d\theta = 3 \cdot 2\pi = 6\pi$; $\theta\big|_0^{2\pi}$; $(L = 2\pi r = 6\pi)$

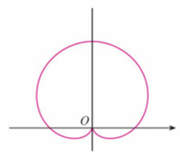
Set-up the integral to

Find the length of the cardioid $r = 1 + \sin\theta$.

$r' = \cos\theta$

Solution:

The cardioid is shown in Figure 8.



$r = 1 + \sin\theta$

Figure 8

$L = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$; $\sin^2\theta + \cos^2\theta = 1$

$= \int_0^{2\pi} \sqrt{2(1+\sin\theta)}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta$

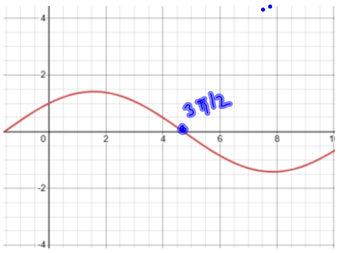
$1 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}$; $\sin 2\theta = 2\sin\theta\cos\theta$; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= \sqrt{2}\int_0^{2\pi} \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\, d\theta$

$a^2 + b^2 + 2ab = (a+b)^2$

Set-up

$= \sqrt{2}\int_0^{2\pi} \sqrt{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}\, d\theta = \boxed{\sqrt{2}\int_0^{2\pi} \left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right| d\theta}$



$$L = \sqrt{2}\left[\int_0^{3\pi/2}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{3}\right)d\theta + \int_{3\pi/2}^{2\pi}\left(-\cos\theta - \sin\frac{\theta}{2}\right)d\theta\right]d\theta$$

(exercise) $\simeq 8$.

~~5.5~~, ~~6.1~~, ~~6.2~~, ~~6.3~~

7.1, 7.2, 7.3, 7.4, 7.8

8.1, ~~8.2~~

10.1, 10.2, 10.3, 10.4 ✓

~~11.1~~, (11.2) ✓, (11.3) ✓

11.2

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by 3

$\{a_n\}$ ✓ or $\{a_n\}_{n=1}^{\infty}$ ✓

EXAMPLE 3 Finding the *n*th Term of a Sequence

Find the *n*th term of each of the following sequences. Assume that the indicated patterns continue.

Solution

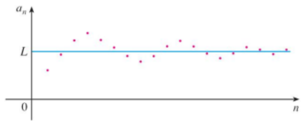
Sequence	*nth* term (general term)
(a) $e, \frac{e^2}{2}, \frac{e^3}{3}, \ldots$ (a_1, a_2, a_3)	$a_n = \frac{e^n}{n}$, $n = 1, 2, 3, \ldots$
(b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ ($b_1, b_2, b_3, b_4, \ldots$)	$b_n = \left(\frac{1}{3}\right)^{n-1}$ ✓
(c) $1, 4, 9, 16, 25, \ldots$	$c_n = n^2$ ✓
(d) $\frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \ldots$	$d_n = \frac{2n}{n+1}$
(e) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$	$e_n = \frac{(-1)^{n+1}}{n}$
(f) $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{6}, \ldots$	$f_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$

---

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).



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3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$. $= \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$.

L'hopital's rule

$\lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$

Find $\lim_{n\to\infty} \frac{n}{n+1}$. $= \lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1} = 1.$

The method is similar to

Using a Related Function to Show a Sequence ...

Show that $\left\{\frac{3n^2+5n-2}{6n^2-6n+5}\right\}$ converges and find its limit.

Solution The function

Calculus 1: $\lim_{n\to\infty} \frac{3n^2+5n-2}{6n^2-6n+5} = \frac{3}{6} = \frac{1}{2}.$

calculus 2: $\lim_{x\to\infty} \frac{3x^2+5x-2}{6x^2-6x+5} = \lim_{x\to\infty} \frac{6x+5}{12x-6}$

$= \lim_{x\to\infty} \frac{6}{12} = \frac{1}{2}.$

Show that $\left\{\frac{n}{e^n}\right\}$ converges and find its limit.

$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$

5 Determine Whether a Sequence Converges or Diverges

A sequence $\{s_n\}$ diverges if $\lim_{n\to\infty} s_n$ does not exist. This can happen if

- there is no single number L that the terms of the sequence approach as $n \to \infty$
- $\lim_{n\to\infty} s_n = \infty$

DEFINITION Divergence of a Sequence to Infinity

The sequence $\{s_n\}$ **diverges to infinity**, that is,

$$\lim_{n\to\infty} s_n = \infty$$

if, given any positive number M, there is a positive integer N so that whenever $n > N$, then $s_n > M$.

EXAMPLE 7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent, $n = 1, 2, 3, \ldots$

$a_n = \{-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots\}$

Show that the following sequences diverge:

(a) $\{1 + (-1)^n\}$ divergent **(b)** $\{n\}$ divergent. $\{1, 2, 3, 4, 5, \ldots\}$ $\lim_{n\to\infty} n = \infty$.

Solution (a) The terms of the sequence are 0, 2, 0, 2 ...

$\{0, 2, 0, 2, 0, 2, 0, 2, \ldots\}$

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 8 Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists. $\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots\}$

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore, by Theorem 6,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

---

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$r = -1 \Rightarrow \{-1\}^n \Rightarrow$ divergent

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$r = 1$ $\{1, 1, 1, 1, 1, 1, 1, \ldots\}$

$r = \frac{1}{2} \Rightarrow \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = \lim_{n\to\infty} \frac{1}{2^n} = 0$

$r = 2 \Rightarrow 2^n = \{2, 4, 8, 16, 32, \ldots\}$

mcq: which one of the following sequences is convergent?

a) $\{(-1)^n\}$ b) $\{(-\frac{1}{2})^n\}$ c) $\{(-\frac{5}{4})^n\}$

d) $\{n\}$ e) $\{(\frac{3}{2})^n\}$

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

1 $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

30
300
3000

If $\lim_{n \to \infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Geometric Series

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

constant

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

$S_n = a + a + a + \cdots + a = a \cdot n$

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$.

$\lim_{n\to\infty} S_n = a\infty = \infty$

Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

21

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

3
$$s_n = \frac{a(1 - r^n)}{1 - r}$$

$\lim_{n\to\infty} S_n = \frac{a}{1-r} \cdot \lim_{n\to\infty} 1 - r^n$ $-1<r<1$

$= \frac{a}{1-r}$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

$-1<r<1$

If $|r| \geqslant 1$, the geometric series is divergent.

$\dfrac{\text{First term}}{1-\text{Common ratio}}$

sum.

(a) $\displaystyle\sum_{n=1}^{\infty} 8\left(\frac{2}{5}\right)^{n-1}$

$a = 8$, $r = 2/5 \Rightarrow -1<r<1$

$= \dfrac{8}{1-\frac{2}{5}} = \dfrac{8}{\frac{3}{5}} = \dfrac{8\cdot 5}{3} = \dfrac{40}{3}.$

This series converges to 40/3

The sum is 40/3

(c) $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n-1}$ $a=3$

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

1 and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

r is $\frac{3}{2}$. is r between -1 and 1. NO. divergent.

Find the sum of the geometric series

$$5 - \tfrac{10}{3} + \tfrac{20}{9} - \tfrac{40}{27} + \cdots$$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

$5 \times ? = -10/3$ ($-2/3$)

$-\frac{10}{3} \times ? = \frac{20}{9}$ ($-2/3$)

$\frac{20}{9} \times ? = -\frac{40}{27}$ ($-2/3$)

Common ratio $= -\frac{2}{3}$, $-1<-\frac{2}{3}<1$

Sum $= \dfrac{\text{First term}}{1-\text{Common ratio}} = \dfrac{5}{1-\left(-\frac{2}{3}\right)} = \dfrac{5}{1+\frac{2}{3}} = \dfrac{5}{\frac{5}{3}} = 3.$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

< 1 and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

etric series is divergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} 1\cdot\left(\frac{1}{2}\right)^n$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

Common ratio $= \frac{1}{2}$, $-1<r<1$

First term $= 1/2$

$\dfrac{\text{First term}}{1-\text{Common ratio}}$

$\dfrac{1/2}{1-1/2} = 1.$

Common mistake

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^n$$

$a = 1$

$r = \frac{1}{2}$

$$\frac{a}{1-r}$$

$$\frac{1}{1-1/2} = 2$$

EXAMPLE 6 Using a Geometric Series with a Bouncing Ball

A ball is dropped from a height of 12 m. Each time it strikes the ground, it bounces back to a height three-fourths the distance from which it fell. Find the total distance traveled

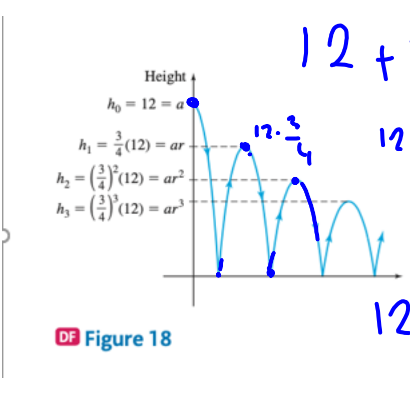


Figure 18

$12 + 2\cdot12\cdot\frac{3}{4} + 2\cdot12\cdot\frac{3}{4}\cdot\frac{3}{4} + \cdots$

$12 + 2\cdot12\cdot\frac{3}{4}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right)$ geometric $a = 1$, $r = \frac{3}{4}$

$12 + 2\cdot12\cdot\frac{3}{4}\cdot\frac{1}{1-\frac{3}{4}}$

$12 + 2\cdot12\cdot\frac{3}{4}\cdot4$

$12 + 72 = 84$ meters.

EXAMPLE 9 Show that the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

$\sum_{n=1}^{\infty}\frac{1}{n}$ is called harmonic series and it is divergent. (Proof will be given in 11.3)

EXAMPLE 8 Show that the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent, and find its sum.

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$

$1 = A(n+1) + B\cdot n$

$n = 0$: $1 = A$

$n = -1$: $1 = -B \Rightarrow B = -1$

S_n = The sum of first n-terms:

$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$S_n = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1 =$ SUM

---

11. $\sum_{k=1}^{\infty}\left(\frac{1}{k+2} - \frac{1}{k+3}\right)$ 1

$S_n = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$

$S_n = \frac{1}{3} - \frac{1}{n+3}$ $\lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{n+3}\right) = \frac{1}{3}$.

---

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (p, q) True

If $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum a_n$ is convergent False

example: $\sum\frac{1}{n}$ (a_n). $\lim_{n\to\infty}\frac{1}{n} = 0$ but $\Rightarrow \sum\frac{1}{n}$ is divergent

if p then q $\Leftrightarrow$ if not q then not p

7 Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

ex: $\sum_{n=1}^{\infty}\frac{8n^2}{7n^2+1}$ (a_n). $\lim_{n\to\infty}\frac{8n^2}{7n^2+1} = \frac{8}{7} \neq 0$

Conclusion: $\sum_{n=1}^{\infty}\frac{8n^2}{7n^2+1}$ is divergent.

---

mcq: Let $\sum_{n=1}^{\infty}\frac{7n^2-2}{5n^2+1}$ be a series.

Which one is true

a) The first term is 0. The first term $= \frac{7-2}{5+1} = \frac{5}{6}$

b) Convergent

c) divergent

d) The sum of first two terms (S_2) is 0 $\frac{5}{6} + \frac{26}{21}$ (First term, Second term)

e) ....

EXAMPLE 11 Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

$\sum_{n=1}^{\infty}\frac{3}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n}$

$3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n}$ $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ common ratio $= \frac{1}{2}$, $a = \frac{1}{2}$

$3\cdot1 + 1$ $\frac{a}{1-r} = \frac{1/2}{1-\frac{1}{2}} = 1$

$= 4$

EXAMPLE 11 Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \left(\frac{3}{2}\right)^n\right)$.

$\sum_{n=1}^{\infty}\frac{3}{n(n+1)} + \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n \rightarrow \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \cdots$ common ratio $= \frac{3}{2}$

Divergent. Because $\frac{3}{2}$ is not in $(-1,1)$

$3 +$ (Divergent) $=$ Divergent

11.3
(integral Test)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Test the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ for convergence or divergence.

$f(x) = \frac{1}{x^2+1}$. f is continous, f is positive, f is decreasing? $f' = \frac{-2x\cdot1}{(x^2+1)^2} < 0$

I can apply integral test.

$\int_1^{\infty}\frac{1}{x^2+1}\,dx$ (7.8)

$\lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\arctan x\Big|_1^t = \lim_{t\to\infty}(\arctan t - \arctan 1)$

$= \lim_{t\to\infty}\arctan t - \frac{\pi}{4}$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$

Since $\int_1^{\infty}\frac{1}{x^2+1}$ is convergent, $\sum\frac{1}{n^2+1}$ is convergent

$\int_1^\infty \frac{1}{x^p}\,dx$ — $p>1$ → convergent; $p \le 1$ → divergent

$\sum_{n=1}^{\infty} \frac{1}{n^p} \rightarrow \begin{cases} p>1, & \text{convergent} \\ p \le 1, & \text{divergent} \end{cases}$ (P series)

($p=1$, harmonic series)

MCQ: which one is convergent

a) $\sum_{n=1}^{\infty} \frac{1}{n^{-3}}$

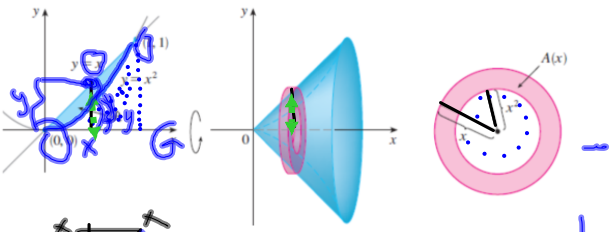
b) $\sum_{n=1}^{\infty} 7n^5$ $\left(\frac{7}{n^{-5}}\right)$

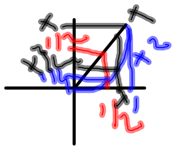
c) $\sum 1907\, n^{-10}$ $\left(\frac{1}{n^{10}}\right)$

d) $\sum_{n=1}^{\infty} \frac{1}{n}$

e) $\sum_{n=1}^{\infty} 1955\, n^{-2/3}$ $\left(\frac{1}{n^{2/3}}\right)$

EXAMPLE 4 The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.





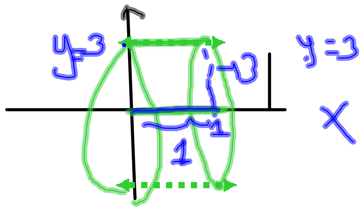
$$V = \pi \int_0^1 x^2\, dx - \pi \int_0^1 (x^2)^2\, dx$$

$$V = \pi \frac{x^3}{3}\Big|_0^1 - \pi \frac{x^5}{5}\Big|_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{15}\pi.$$

---

ex. $y = 3$, $0 < x < 1$. Rotate x.



$$V = \pi \int_0^1 r^2\, dx = \pi \int_0^1 3^2\, dx$$

$$= 9\pi\, x\Big|_0^1$$

$$= 9\pi.$$

Geometry

$$V = \pi r^2 h = \pi \cdot 3^2 \cdot 1$$

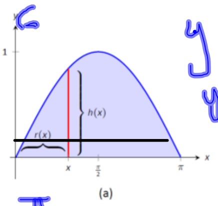
$$= 9\pi$$

6.3

> **2** The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is
>
> $$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{where } 0 \leqslant a < b$$

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution



$x_2 = \arcsin y$

$x_2 = x_1 - \pi$

$= \arcsin y$

$\Rightarrow x_1 = \arcsin y + \pi$

$y = \sin(x_1)$, $x_1 - \pi$

$y = \sin(x_2)$

$\sin(x_1) = \sin(x_2)$

$x_2 = x_1 - \pi$

$\text{Disk} = \pi \int_0^1 x_2^2\, dy - \pi \int_0^1 x_1^2\, dy$

$= \pi \int_0^1 \arcsin^2 y\, dy - \pi \int_0^1 (\arcsin y + \pi)^2 dy$

hard!!!

Shell:

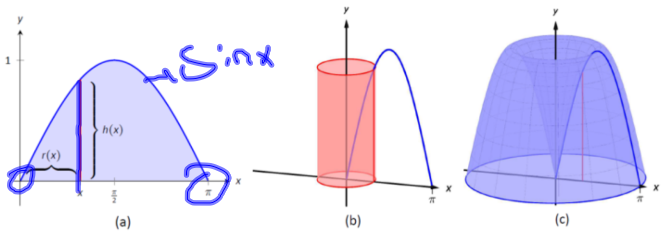


Figure 6.3.6a: Graphing a region in Example 6.3.4

$$V = 2\pi \int_0^\pi x \sin x\, dx.$$

Integration by parts. We solve this integral in 7.1

his requires Integration By Parts. Set $u = x$ and $dv = \sin x\, dx$; we leave it to the reader to fill in the rest. We have:

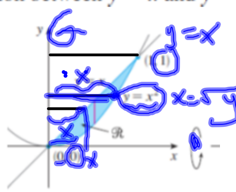
$$= 2\pi\Big[-x\cos x\Big|_0^\pi + \int_0^\pi \cos x\, dx\Big]$$

$$= 2\pi\Big[\pi + \sin x\Big|_0^\pi\Big]$$

$$= 2\pi\Big[\pi + 0\Big]$$

$$= 2\pi^2 \approx 19.74 \text{ units}^3.$$

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



Disk:

$\pi \int_0^1 (\sqrt{y})^2 dy - \pi \int_0^1 (y)^2 dy$

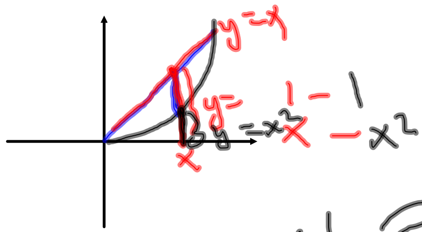
$\pi \left[\frac{y^2}{2}\Big|_0^1 - \frac{y^3}{3}\right]$

$\pi\left[\frac{1}{2} - \frac{1}{3}\right] = \pi \cdot \frac{1}{6}$

Shell!

Shell

$2\pi \int_0^1 x f(x)\, dx$ (height)



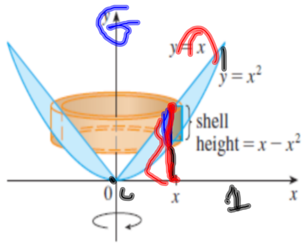


FIGURE 8

$1 = 1 - 1$

$x - x^2$

$\pi = x - x^2$

$2\pi \int_0^1 x(x - x^2)\, dx$

$x^2 - x^3$

$2\pi \cdot \frac{1}{12} = \pi/6$

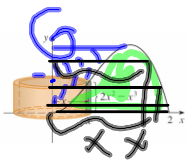
$\frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1$

$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

Find the volume of the solid obtained by rotating about the *y*-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Solution:

From the sketch in Figure 6 we see that a typical shell has radius *x*, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$.



$\pi \int x^2\, dy - \pi \int \ldots$

$y = 2x^2 - x^3$

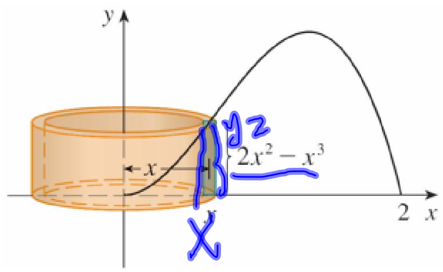
$x = ?$

$x^2 = ?$

Disk: Terrible

Shell:

$$2\pi \int_0^2 x\left(2x^2 - x^3\right)dx$$



$$2x^3 - x^4$$

$$= \frac{2x^4}{4_2} - \frac{x^5}{5}$$

$$= 8 - \frac{32}{5}$$

$$= 8/5$$

$$2\pi \cdot \frac{8}{5} = \frac{16\pi}{5}$$

Quiz

5.5, 6.1, 6.2, 6.3

10 + 1 = 11 pts.
(1 = Bonus)